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Some Applications of Conformal Transformation to Problems in Hydrodynamics

J. G. Leathem

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XIV. *Some Applications of Conformal Transformation to Problems in Hydrodynamics.*

By J. G. Leathem, *M.A.*, *D.Sc.*

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Introduction.

1. The problem of the conformal representation of the part of the plane of a variable z, which is bounded by a rectilineal polygon, upon the half-plane of a variable w bounded by the real axis, is solved (save for an integration) by the well-known transformation of Schwarz

$$dz = \mathrm{C}\Pi\,(w-a_r)^{-\alpha_r/\pi}\,dw,$$

where C, a_1, a_2, &c., are real constants, and $\pi-\alpha_1$, $\pi-\alpha_2$, &c., are the internal angles of the rectilineal polygon.

A more difficult problem is that of the conformal representation upon the half-plane of w of a region in the z plane whose boundary is partly curved; it is with this problem that the present paper is concerned, always however with a view to interpretation of results in terms of the two-dimensional flow of liquid in regions having particular types of boundary.

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At the outset something of a suggestion might seem to be found in consideration of the fact that, if the intrinsic equation of a curve in the z plane is $\chi = \mathrm{F}(s)$, the vectorial element of arc dz is the same as $ds \exp(i\chi)$ or $ds \exp\{i\mathrm{F}(s)\}$, where i as usual represents $\sqrt{(-1)}$. From this it appears that the transformation

$$dz = f'(w) \exp[i\mathrm{F}\{f(w)\}]\, dw,$$

where F and f are functions having real values for all real values of their arguments, makes the real axis in the w plane correspond to a curve in the z plane whose intrinsic equation is $\chi = \mathrm{F}(s)$, the relation between w and s being $s = f(w)$, where f is an arbitrary function. But transformations of this type, though they settle the correspondence of prescribed boundaries, offer no guarantee against zeros, infinities or singularities at points where w is not real, and so do not necessarily give conformal representation. The geometrical idea of the transformations is nevertheless useful.

2. Some problems of conformal transformation with partially curvilinear boundaries were worked out by Mr. W. M. PAGE in a paper* published by the London Mathematical Society a few years ago. Mr. PAGE definitely rejects, on account of an apparent indeterminateness, the method of treating a curve as the limit of a rectilineal polygon and seeking the limit of the product of the corresponding Schwarzian factors. Instead he has recourse to the empirical but useful device of writing down the Schwarz transformation as it would be for a rectilineal polygon having the same angles as the prescribed curvilinear one, and then introducing into the expression for dz/dw a further factor which has the effect of making the relevant side of the z polygon curvilinear while leaving the other sides straight. There is no question of prescribing the form of the curve in the z plane, one simply takes such conformal transformations as one can succeed in formulating in the above manner and tries to find out what sorts of curve they yield.

Mr. PAGE, acknowledging a suggestion by Mr. H. W. RICHMOND, takes factors which occur in well-known problems, and by associating them with new sets of Schwarzian factors obtains a new series of results. The familiar case of a semi-circular boss on a straight line, namely

$$dz = \mathrm{C}\,(w^2-c^2)^{-1/2}\,\{w+(w^2-c^2)^{1/2}\}\, dw, \quad \ldots \quad (1)$$

supplies the factor $w+(w^2-c^2)^{1/2}$, and the case of a semi-elliptic boss on a straight line, namely

$$dz = \mathrm{C}\,(w^2-c^2)^{1/2}\,\{w \sinh \alpha + (w^2-c^2)^{1/2} \cosh \alpha\}, \quad \ldots \quad (2)$$

supplies the factor $w \sinh \alpha + (w^2-c^2)^{1/2} \cosh \alpha$. These two factors and their square roots are the factors which Mr. PAGE associates with other factors of the Schwarzian type in order to obtain new results.

* "Some Two Dimensional Problems in Electrostatics and Hydrodynamics," 'Proc. Lond. Math. Soc., Ser. 2, vol. XI., 1912–13, p. 313.

3. One object of the present paper is to try to extend the range of transformations available for the conformal representation upon a half-plane of a region bounded by a polygon some of whose sides are straight and some curvilinear. Aspects of the subject will be discussed which have the theoretical advantage of being to some extent free from the empirical character of Mr. PAGE's method, but the application of these to particular cases presents great practical difficulty. It is therefore proposed to begin by studying those functions of w which, when introduced as factors into formulæ of transformation which are otherwise of the Schwarzian type, lead to conformal representations of the required general character. Every increase in the range of such functions made available, and every advance in knowledge of the theory of such functions, diminishes the empirical element in their employment for the solution of particular problems.

The main object of the paper is the extension of the range of solvable problems in two-dimensional liquid motion, and cases of such motion will be used as illustrations of the pure mathematical theory.

CONFORMAL CURVE-FACTORS.

4. *Definition and Characteristics.*—Consider a transformation

$$dz = \mathrm{A}\mathscr{C}\Pi\,(w-a_r)^{-\alpha_r/\pi}\,dw, \qquad (3)$$

wherein A, α_1, α_2, ... are real constants, a_1, a_2, ... real constants in ascending order of magnitude, and $\mathscr{C}$ a function of w. If $\mathscr{C}$ were absent this would be a Schwarzian transformation giving a conformal representation upon the half-plane of w of the region in the z plane bounded by a rectilineal polygon of external angles α_1, α_2, α_3 .... Suppose $\mathscr{C}$ to be such that the transformation as it stands gives a conformal representation upon the half-plane of w of the region in the z plane bounded by a polygon having the external angles α_1, α_2, α_3, ... as before, all its sides but one straight, and that one side (say the side corresponding to $a_{r+1} > w > a_r$) curvilinear. A function $\mathscr{C}$ of w having this property may be called a "conformal curve-factor," or, for brevity, a "curve-factor." In view of the possibility of our having to consider functions which, when thus introduced into a Schwarzian transformation without spoiling its conformal character, replace the rectilineal polygon by a polygon having the same angles and not one only but several curved sides, it may be well to distinguish between "simple," "double," "triple," &c. conformal curve-factors, according to the number of curved sides which they introduce into the polygon.

If a function $\mathscr{C}$ of w is a simple conformal curve-factor it must satisfy the following requirements :—

(i.) The vector angle of the complex $\mathscr{C}$ must have a constant value for all real values of w greater than a_{r+1}, and a constant value for all real values of w less than

a_r. For real values of w intermediate between a_r and a_{r+1} the vector angle of $\mathscr{C}$ must change continuously with w.

(ii.) $\mathscr{C}$ must not be zero or infinite for any definite value of w which is real, or for any definite complex value of w corresponding to a point on the relevant (positive) side of the axis of w real. Such zeros or infinities would destroy the conformal character of the transformation, and would, if occurring on the boundary, interfere with the prescribed arrangement of corners. The word "definite" is here used so that the restriction may be understood not to apply to w infinite. Certain singularities at definite points might be capable of interesting interpretations in the application to hydrodynamics, but it seems best at present to exclude them.

(iii.) $\mathscr{C}$ must not have any definite branch points on the relevant side of the axis of w real. It may however have branch points on the axis of w real, since this axis serves as a barrier preventing such circulation round a branch point in it as would make the function many-valued.

(iv.) The form of $\mathscr{C}$ for w infinite must conform to conditions depending on the nature of the particular problem to which the transformation is to be applied.

5. *Linear and Angular Ranges.*—The range on the axis of w real corresponding to values of w for which the vector angle of $\mathscr{C}$ is variable may be called the "linear range" of the curve-factor. The difference between the vector angles at the extremities of the linear range may be called the "angular range" of the curve-factor, this being reckoned as positive when the greater angle corresponds to the smaller value of w.

In the case of a simple curve-factor the linear range consists of the real values of w between those typified by a_r and a_{r+1}, and the angular range is the angle between the tangents at the extremities of the corresponding curved side of the polygon, the standard case being that of convexity towards the relevant region.

A curve-factor may of course have its angular range zero; in that case there will be an inflexion on the corresponding curved side of the polygon.

It is readily seen that if $\mathscr{C}$ be a curve-factor, and if n be a constant, then $\mathscr{C}^n$ also is a curve-factor, having the same linear range as $\mathscr{C}$, but its angular range equal to n times that of $\mathscr{C}$. It is important to notice that this statement is valid even if n is negative, for as $\mathscr{C}$ has no zeros or infinities in the relevant region (except possibly for w infinite), the raising of $\mathscr{C}$ to a negative power does not introduce any fresh infinities or zeros. But if $\mathscr{C}$ has a positive angular range and if n is negative the angular range of $\mathscr{C}^n$ is negative. Thus, if $\mathscr{C}$ gives a curve which is everywhere convex to the relevant region, positive powers of $\mathscr{C}$ give curves having the same characteristic, but negative powers of $\mathscr{C}$ give curves which are concave to the relevant region.

In connexion with requirement (iv.) above, the form of $\mathscr{C}$ for w infinite is sometimes important. If, for $w \to \infty$, $\mathscr{C} \to kw^m$ where k is a constant, it will be convenient to speak of m as the "order at infinity" of the curve-factor. Clearly there may be curve-factors which do not possess an order at infinity in this sense.

It is important to bear in mind that the form of the curved side of the polygon which corresponds to a particular curve-factor depends not merely on the analytical form of the curve-factor, but also on the Schwarzian or other factors which appear in the formula of transformation. Thus the same curve-factor leads to different curves in different transformations. But these curves will have some features in common, provided the linear range with which the particular curve-factor is associated is not also in whole or in part included in the range of some other curve-factor; for example if $\mathscr{C}$ in any such simple connexion represents a curve everywhere convex to the relevant region, then in other such connexions both $\mathscr{C}$ and positive powers of $\mathscr{C}$ will always represent curves convex to the relevant region. This appears from consideration of the vector angle of $\mathscr{C}$, which, when added to the vector angles of the other factors, the latter being constants for the linear range of $\mathscr{C}$, gives the angle of direction of the tangent to the curve.

6. *Convention as to Fractional Powers.*—In the course of the work expressions involving square roots and other fractional powers of quadratic expressions in w will occur frequently. The convention must therefore be made at the outset that when λ is fractional an expression such as $(w-a)^{\lambda}$ shall be interpreted as having a positive real value for real values of w greater than a, and for other values of w corresponding to points on the relevant (positive) side of the axis of w real such value as corresponds to continuous passage from a point where w is real and greater than a without departure from the relevant half-plane, the branch point $w = a$ being avoided when necessary by a detour in the relevant region. With this convention an expression such as $(w-a)^{\lambda}$ will be free from ambiguity, and in particular the passage round the point a from a real value greater than a to a real value less than a will lead to the form $(a-w)^{\lambda}\exp(i\lambda\pi)$ for real values of w less than a.

7. *The Curve-factor of Semi-circular Type.*—It is natural to expect that the study of the two curve-factors employed by Mr. PAGE may lead to suggestions for the construction of other curve-factors. That which occurs in the transformation (1) above may be called the curve-factor of semi-circular type; it is

$$\mathscr{C}_1 = w+(w^2-c^2)^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and has linear range from $+c$ to $-c$, and angular range π.

It is to be noted that the vector angle of $\mathscr{C}_1$, when $c > w > -c$, is $\tan^{-1}[(c^2-w^2)^{1/2}/w]$, and it is only because the denominator of the fraction in the square bracket vanishes for a point in the linear range that the angular range is π instead of being zero.

The reason why $\mathscr{C}_1$ satisfies the requirement of having no zero is easy to see. Let $\mathscr{D}_1$ be the surd conjugate to $\mathscr{C}_1$, that is $w-(w^2-c^2)^{1/2}$. Every zero of $\mathscr{C}_1$ must (since $\mathscr{D}_1$ has no infinities) be a zero of the product $\mathscr{C}_1\mathscr{D}_1$, but $\mathscr{C}_1\mathscr{D}_1 = c^2$ and has no zeros, therefore $\mathscr{C}_1$ has no zeros.

If the attempt be made to construct other curve-factors of the type of a rational function plus a square root, it appears that for a simple curve-factor the root sign must cover only a quadratic in w, with real factors. Other factors, if imaginary, would introduce branch-points; if real they would introduce fresh corners of angle π, and so effectively yield a multiple curve-factor instead of a simple one. Thus the type must be

$$f(w)+g(w)(w^2-c^2)^{1/2},$$

where f and g are rational functions. If this is to be free from zeros in virtue of neither it nor its conjugate surd having zeros, it is necessary that

$$\{f(w)\}^2-\{g(w)\}^2(w^2-c^2) = \text{const.}$$

If $g(w)$ is of the first order the rationality of $f(w)$ requires $g(w)$ to be simply w, and then the curve-factor is

$$\mathscr{C}_2 = w^2-\tfrac{1}{2}c^2+w(w^2-c^2)^{1/2}, \qquad (5)$$

which has the angular range 2π. This is not really a new result, for $\mathscr{C}_2 = \frac{1}{2}\mathscr{C}_1^2$. Similarly, if $g(w)$ be assumed of the second order, the rationality of $f(w)$ restricts to the form

$$\mathscr{C}_3 = w^3-\tfrac{3}{4}c^2w+(w^2-\tfrac{1}{4}c^2)(w^2-c^2)^{1/2}, \qquad (6)$$

which is not a new result, since $\mathscr{C}_3 = \frac{1}{4}\mathscr{C}_1^3$.

It seems therefore safe to conclude that the semi-circular curve-factor is a very special type which does not admit of generalisation.

8. *The Curve-factor of Semi-elliptic Type.*—The factor which occurs in transformation (2) above may be called the curve-factor of semi-elliptic type. It is

$$\mathscr{C}_4 = w\sinh\alpha+(w^2-c^2)^{1/2}\cosh\alpha. \qquad (7)$$

This has, for $c > w > -c$, the vector angle $\tan^{-1}[(c^2-w^2)^{1/2}\cosh\alpha/w\sinh\alpha]$, and its angular range is π because the rational part of $\mathscr{C}_4$ vanishes and changes sign at a point in the linear range.

$\mathscr{C}_4$ is free from zeros, but the reason of this is different from that which holds in the case of $\mathscr{C}_1$. Denoting the surd conjugate to $\mathscr{C}_4$ by $\mathscr{D}_4$, it is seen that

$$\mathscr{C}_4\mathscr{D}_4 = c^2\cosh^2\alpha-w^2, \qquad (8)$$

and this has two real zeros. It has to be shown that both these zeros belong to $\mathscr{D}_4$, so that $\mathscr{C}_4$ has none.

For w real $\mathscr{C}_4$ has three different forms, namely

$$\begin{aligned} &(w > c), && w\sinh\alpha+(w^2-c^2)^{1/2}\cosh\alpha, \\ &(c > w > -c), && w\sinh\alpha+i(c^2-w^2)^{1/2}\cosh\alpha, \\ &(-c > w), && w\sinh\alpha-(w^2-c^2)^{1/2}\cosh\alpha. \end{aligned}$$

The first of these is the sum of two real positive quantities, and so cannot vanish; the second is a complex whose real and imaginary parts have no common factor, and so cannot vanish; the third, because the real part has changed sign in the linear range, is the sum of two real negative quantities, and so cannot vanish. Thus $\mathscr{C}_4$ has no real zeros, and accordingly, as no imaginary zeros are possible, has no zeros at all. The change of sign of the real part of $\mathscr{C}_4$ is an important element in the argument.

9. *Modified Semi-elliptic and Semi-circular Types.*—Within the limits of those properties by which the freedom of $\mathscr{C}_4$ from zeros is secured, namely that the product of the curve-factor and its conjugate surd has only real roots and that both terms of the curve-factor have the same sign for real values of w outside the linear range, there is room for modification of the type $\mathscr{C}_4$ by the introduction of an additional parameter. Considering

$$\mathscr{C}_5 = (w-k)\sinh\alpha + (w^2-c^2)^{1/2}\cosh\alpha, \quad \ldots \quad (9)$$

it is seen that, provided $c > k > -c$, both terms of $\mathscr{C}_5$ are positive for real values of w greater than c, and both negative for real values of w less than $-c$; thus $\mathscr{C}_5$ cannot vanish for any real value of w. If the conjugate surd be $\mathscr{D}_5$,

$$\mathscr{C}_5\mathscr{D}_5 = (c^2+k^2\sinh^2\alpha)\cosh^2\alpha - (w+k\sinh^2\alpha)^2, \quad \ldots \quad (10)$$

which has only real zeros. Hence $\mathscr{C}_5$ has no zeros, and is a curve-factor. Its angular range is π. On the other hand, if $k > c$, $\mathscr{C}_5$ is positive for $w = +\infty$ and negative for $w = c$, so it must have a zero and is not a curve-factor. The same applies when $k < -c$, since then $\mathscr{C}_5$ is negative for $w = -\infty$ and positive for $w = -c$.

Similarly $\mathscr{C}_1$ may be modified to the form

$$\mathscr{C}_6 = w-k+(w^2-c^2)^{1/2} \quad \ldots \quad (11)$$

with the restriction $c > k > -c$. The only zeros that might be possible are those of

$$\mathscr{C}_6\mathscr{D}_6 = -2kw+k^2+c^2, \quad \ldots \quad (12)$$

and this has only one zero, which is real. Thus $\mathscr{C}_6$ has no zeros, and is a curve-factor of angular range π.

It may be noted that

$$\mathscr{C}_6 = \lim_{\alpha\to\infty} e^{-\alpha}\mathscr{C}_5. \quad \ldots \quad (13)$$

The utility of the adjustable parameter k will be seen in the working of particular examples.

10. The form of $\mathscr{C}_5$ suggests another possible curve-factor,

$$\mathscr{C}_7 = (w-k)\cosh\alpha + (w^2-c^2)^{1/2}\sinh\alpha. \quad \ldots \ldots \quad (14)$$

Any zeros which this may have must be zeros of

$$\mathscr{C}_7\mathscr{D}_7 = (w-k\cosh^2\alpha)^2 + \sinh^2\alpha\,(c^2-k^2\cosh^2\alpha). \quad \ldots \ldots \quad (15)$$

If $k^2\cosh^2\alpha > c^2$, the zeros of $\mathscr{C}_7\mathscr{D}_7$ are both real, and if $c^2 > k^2$, $\mathscr{C}_7$ cannot have real zeros. Hence $\mathscr{C}_7$ is a curve-factor if $c^2 > k^2 > c^2\operatorname{sech}^2\alpha$.

If $k^2\cosh^2\alpha < c^2$, which involves *à fortiori* that $k^2 < c^2$, then $\mathscr{C}_7\mathscr{D}_7$ has imaginary zeros, and the question arises whether $\mathscr{C}_7$ can have any imaginary zeros at points on the relevant side of the axis of w real. This may be tested by means of the theorem that, if a function $f(w)$ be free from infinities in a given region, the sum of the orders of all its zeros in that region is equal to $(1/2\pi i)\int f'(w)/f(w)\,dw$ taken round the boundary; a simpler enunciation of the theorem is that the sum of the orders of all the zeros of $f(w)$ in the region is $\Theta/2\pi$, where Θ is the algebraic sum of all the changes, abrupt or gradual, which take place in the vector angle of $f(w)$ as w makes a complete circuit of the boundary in the positive sense.

In the present application $\mathscr{C}_7$ is put for $f(w)$, the region is the half-plane of w on the positive side of the real axis, and the boundary consists, in the main, of the whole of the real axis and a semi-circle of radius R which tends to indefinite greatness. The points $w = \pm c$, being branch points, have to be avoided by semi-circular detours of infinitesimal radius on the positive side of the axis. If there were zeros on the axis they would have to be avoided by similar detours, but in this case there are none. The points $w = \pm c$ not being zeros or infinities, the detours round them do not result in any alteration in the vector angle of $\mathscr{C}_7$. From $-\infty$ to $-c$ this vector angle is π; from $-c$ to $+c$ it continually diminishes (because $c > k > -c$) from π to zero; from $+c$ to $+\infty$ it is zero; along the semi-circle from $+\infty$ round to $-\infty$ it increases from zero to π. Hence $\Theta = -\pi+\pi = 0$, and so $\mathscr{C}_7$ has no zeros in the relevant half-plane.

Therefore, if $k^2 < c^2\operatorname{sech}^2\alpha$, $\mathscr{C}_7$ is a curve-factor.

11. *Relation between Angular Range and Order at Infinity.*—In the test for the presence or absence of imaginary zeros employed in the previous article it will be noticed that the sum Θ is made up of two parts, one corresponding to the linear range and equal to minus the angular range, the other corresponding to the infinite semi-circle and equal to π multiplied by the order at infinity of the curve-factor. From this it is seen that if an expression of the type

$$f(w)+g(w)\,(w^2-c^2)^{1/2} \quad \ldots \ldots \ldots \quad (16)$$

is free from infinities and real zeros, and has its angular range equal to π times its order at infinity, it will also be free from imaginary zeros in the relevant region, and is a conformal curve-factor.

For example consider

$$\mathscr{C}_8 = A(w-k_1)(w-k_2)+B(w-l)(w^2-c^2)^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $A > 0$, $B > 0$, $c > k_1 > l > k_2 > -c$.

For w real and greater than c, or for w real and less than $-c$, $\mathscr{C}_8$ consists of the sum of two positive quantities. Hence $\mathscr{C}_8$ has no real zeros. The angular range of $\mathscr{C}_8$ is 2π, the vector angle being zero for $w \geqq c$, increasing, as w diminishes, to $\frac{1}{2}\pi$ for $w = k_1$, then to π for $w = l$, then to $\frac{3}{2}\pi$ for $w = k_2$, then to 2π for $w = -c$, and remaining at 2π for $w < -c$. The order at infinity is 2, so that $\Theta = 0$. Hence $\mathscr{C}_8$ has no zeros in the relevant half-plane, and is a conformal curve-factor.

If the angular range were different from π times the order at infinity Θ would not be zero, and the function would not be free from zeros. Thus if in $\mathscr{C}_8$ the order of magnitude of the constants were $c > l > k_1 > k_2 > -c$ the angular range would be zero and $\mathscr{C}_8$ would not be a curve-factor.

It may therefore be taken as an established theorem for all curve-factors having a definite order at infinity that the angular range is equal to π times the order at infinity.

12. *Curve-factors having Branch Points of Unequal Orders at the Extremities of the Linear Range.*—Consider the function

$$\mathscr{C}_9 = A(w-k)+B(w-a)^{\alpha}(w-b)^{1-\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where $A > 0$, $B > 0$, $a > k > b$, and $1 > \alpha > 0$.

For w real and greater than a this is real; for w real and between a and b it takes the form

$$A(w-k)+B(a-w)^{\alpha}(w-b)^{1-\alpha}\exp(i\alpha\pi), \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

having a vector angle which is zero for $w = a$ and increases to π for $w = b$; for w real and less than b it takes the form

$$-A(k-w)-B(a-w)^{\alpha}(b-w)^{1-\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Thus if $\mathscr{C}_9$ is a curve-factor it has linear range a to b, and angular range π. Since both terms of (18) are positive, both terms of (20) negative, and the terms of (19) one real and one complex, $\mathscr{C}_9$ has not any real zeros. Since the angular range is π and the order at infinity unity, the theorem of the previous article shows that $\mathscr{C}_9$ has no imaginary zeros in the relevant half-plane. Hence $\mathscr{C}_9$ is a conformal curve-factor.

Hydrodynamical Illustrations.

13. The utility of the conformal curve-factors so far considered may be exemplified by employing them in the specification of some cases of two-dimensional liquid motion. In such applications the interpretation of w will be that implied by the relation

$$w = \phi+i\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

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where ϕ and ψ are the velocity-potential and stream-function of the liquid motion, so that the velocity components u, v are (when $z = x+iy$)

$$u = -\partial\phi/\partial x = -\partial\psi/\partial y, \qquad v = -\partial\phi/\partial y = \partial\psi/\partial x. \quad . \quad . \quad . \quad . \quad (22)$$

With this interpretation it is known that

$$dz/dw = -q^{-1}\exp(i\chi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where q is the resultant velocity, and χ the angle which its direction makes with the axis of x.

When a transformation is such that for w real and tending to $+\infty$ there is a definite limit for dz/dw, it will be convenient to denote this limit by V^{-1}. This means that the limiting velocity at points indefinitely distant in the direction of the axis of x is a velocity V parallel to the negative direction of that axis.

The interpretation of other constants in the transformation formulæ might be emphasised by explicitly indicating a factor V in every constant which is associated with w by addition or subtraction; for example $(w-c)^{1/2}$ might be replaced by $(w-c'V)^{1/2}$, where c' has the dimensions of a length and depends only on the geometrical configuration in the z plane. But, in order to avoid an unnecessary appearance of heaviness in the formulæ, this device for emphasis will not be employed; it will be easy and sufficient to remember that the velocity of flow corresponding to a given transformation can be altered everywhere in the ratio of V′ to V, without changing the geometrical configuration, by replacing V where it occurs explicitly by V′, and c, in an expression such as $(w-c)^{1/2}$, by cV'/V.

14. *Flow round a Semi-infinite Barrier in the Form of a Wedge with Smoothly Rounded Apex.*—A transformation in which a curve-factor is not accompanied by any Schwarzian factor is typified by

$$dz = C\mathscr{C}^n\,dw, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where C is a constant and $\mathscr{C}$ a curve-factor of definite linear range (say $-c$ to c) and definite angular range γ. For example $\mathscr{C}$ may be any one of the curve-factors already enumerated.

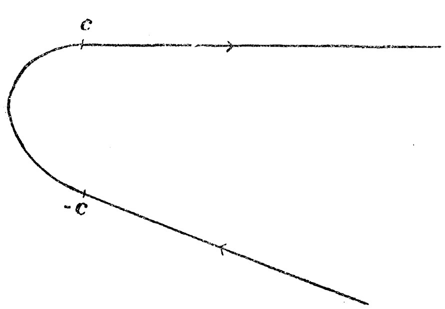

Fig. 1.

The boundary in the z plane corresponding to w real is free from corners. As the point w moves along the negative direction of its real axis the point z moves backwards parallel to the axis of x until w reaches the value c; then z describes a curve whose tangent (with the direction corresponding to w increasing) makes a continually increasing angle with the axis of x until that angle attains the value $n\gamma$; after that, for w still decreasing, the boundary is straight. The configuration is of the general character of fig. 1, wherein the arrows indicate the direction of w increasing, and the relevant region is that on the left of the arrows. The angle $n\gamma$ must be positive since that is π times the order at infinity, and a negative order at infinity would imply infinite velocity at infinity;* the angle $n\gamma$ must not exceed π as otherwise the boundary would intersect itself.

The form and dimensions of the curved part of the boundary depend on the form of $\mathscr{C}$ and on the constants C and n. By equating dx and $i\,dy$ respectively to the real and the imaginary parts of $C\mathscr{C}^n dw$, when w is real and between $+c$ and $-c$, the problem of expressing x and y in terms of a real parameter w is reduced to one of quadrature. The degree of arbitrariness in the form of the curve corresponds to the extent of the range of known forms of $\mathscr{C}$ and the adjustable parameters which these contain.

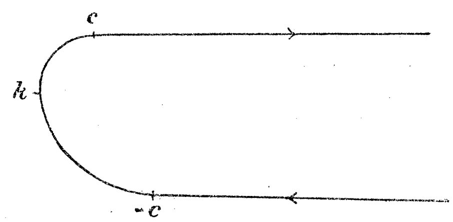

Fig. 2.

A simple case is got by taking $n = 1$, and giving to $\mathscr{C}$ a form $\mathscr{C}_{10}$ representative of $\mathscr{C}_5$ and $\mathscr{C}_7$; this leads to

$$dz = \{A(w-k)+B(w^2-c^2)^{1/2}\}\,dw, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where

$$A > 0,\ B > 0, \text{ and } c > k > -c.$$

The boundary in the z plane consists of two parallel lines extending to infinity on the right, and smoothly joined by a curve on the left, as in fig. 2, the relevant region being to the left of the arrows. Here z can be integrated, and if the origin of z be taken at the point on the boundary where $w = k$, the result is

$$z = \tfrac{1}{2}A(w-k)^2 + \tfrac{1}{2}B\,[w(w^2-c^2)^{1/2} - k(k^2-c^2)^{1/2}$$
$$-c^2\log\{w+(w^2-c^2)^{1/2}\} + c^2\log\{k+(k^2-c^2)^{1/2}\}], \quad . \quad . \quad . \quad (26)$$

whence, for points on the curved part of the boundary,

$$\left.\begin{aligned} x &= \tfrac{1}{2}A(w-k)^2, \\ y &= \tfrac{1}{2}B\,[w(c^2-w^2)^{1/2} - k(c^2-k^2)^{1/2} - c^2\cos^{-1}(w/c) + c^2\cos^{-1}(k/c)]. \end{aligned}\right\} \quad . \quad . \quad (27)$$

The co-ordinates of the extremities of the curve are readily deduced, and it is seen that the distance between the straight parts of the boundary is $\tfrac{1}{2}Bc^2\pi$.

If v_0 be the velocity of the liquid at the point $z = 0$, v_0^{-1} is the value of $|dz/dw|$ for $w = k$. Hence $v_0^{-1} = B(c^2-k^2)^{1/2}$, and the distance between the straight parts of the boundary may be expressed in the form $\tfrac{1}{2}\pi c^2 v_0^{-1}(c^2-k^2)^{-1/2}$. In checking the dimensions of this expression it should be remembered that c and k are both of the dimensions of a length multiplied by a velocity.

* See, however, § 42, *infra*.

15. *Longitudinal Motion of a Ship with Curved Sides terminating in a Pointed Bow and Stern.*—The problem now to be considered is that of the disturbance produced in a liquid stream, which if undisturbed would be uniform, by a stationary object which is bounded by two curves and is symmetrical about a line parallel to the stream. The curves meet at one end at an angle $2p\pi$ and at the other end at an angle $2q\pi$, p and q being each less than a half. The configuration in the z plane is of the general character indicated in fig. 3.

As there is a line of symmetry which is obviously a stream line, only half the configuration need be dealt with, and a transformation is required which will make the boundary shown by a thick line in the z diagram correspond to the axis of w real, and conformally represent the region to the left of the arrows.

The angles in the boundary must be taken account of by Schwarzian factors in the transformation; if c and $-c$ be the values assigned to w at the corners, these factors are $(w-c)^{-p}$ and $(w+c)^{-q}$. The curved side must be represented by a curve-factor $\mathscr{C}$ of linear range c to $-c$.

Let the angular range of a transformation be defined as the angle between the limiting directions of the vector dz for w real and $\to +\infty$ and for w real and $\to -\infty$,

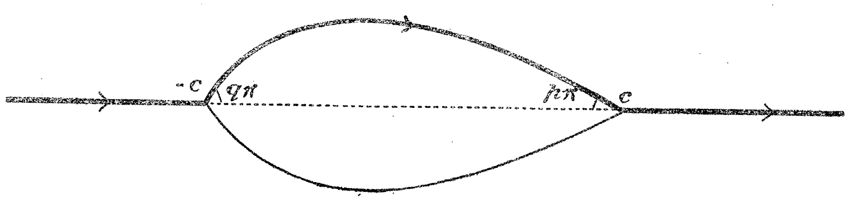

Fig. 3.

measured in the positive sense from the former to the latter. In the present instance the angular range of the transformation is clearly zero. But, if γ be the angular range of the curve-factor, the angular range of the transformation is made up of $-p\pi$ contributed by one Schwarzian factor, $-q\pi$ contributed by the other, and γ contributed by the curve-factor. Hence $\gamma-(p+q)\pi = 0$, and the curve-factor must have an angular range of $(p+q)\pi$. This can be secured by taking the $(p+q)^{\text{th}}$ power of a curve-factor of angular range π. Thus the type of transformation is

$$dz = \mathrm{C}\,(w-c)^{-p}(w+c)^{-q}\,\mathscr{C}^{(p+q)}\,dw, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where $\mathscr{C}$ has the linear range c to $-c$, and an angular range π.

Of the curve-factors so far considered the most general is $\mathscr{C}_9$. The use of this, in a slightly modified form, leads to the transformation

$$dz = \frac{\{\lambda(w-k)+(w-c)^{a}(w+c)^{1-a}\}^{(p+q)}\,dw}{\mathrm{V}\,(\lambda+1)^{p+q}(w-c)^{p}(w+c)^{q}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where $\lambda > 0$, $c > k > -c$, and it has been arranged that $|dz/dw| \to \mathrm{V}^{-1}$ for $w \to +\infty$.

It must be remembered, however, that this transformation has been constructed solely with a view to the angular configuration, and that it affords no guarantee that the stream-lines corresponding respectively to $w > c$ and $w < -c$ will not be merely parallel instead of being in the same straight line.

There is, in fact, the possibility of a configuration of the character indicated in fig. 4, and it must now be shown that this possibility can be provided against by a suitable adjustment of the parameter k.

16. *Significance of the Early Terms in the Expansion of w for Great Values of z.*—If in a transformation such as that of formula (29) the variable w be supposed to have its modulus large, and an expansion be made in ascending negative powers of w, the result is a formula of the type

$$dz = V^{-1}\{1+Sw^{-1}+VDw^{-2}\}\,dw, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

where negative powers of w beyond the second are neglected, and S and D are constants.

Omitting in the first instance the term in w^{-2}, it is seen that the first approxi-

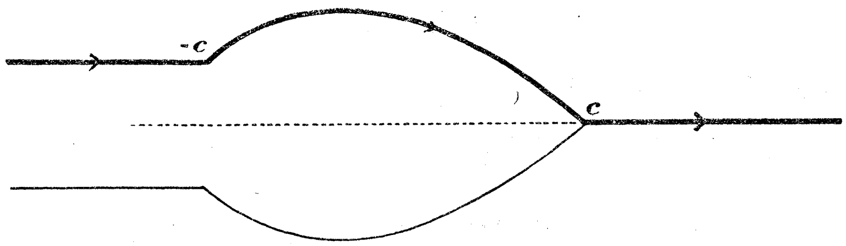

Fig. 4.

mation to z is $V^{-1}w$, and that to the next degree of approximation the formula is equivalent to

$$dw = (V-Sz^{-1})\,dz,$$

whence

$$w = Vz - S\log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

In this expression for w the first term represents uniform flow V in the negative direction of the real axis, and the second term represents a source of strength S at the origin. If there be superposed a uniform motion V in the positive sense of the real axis there results a liquid motion due to the uniform motion of the internal boundary with velocity V, the liquid being "at rest at infinity"; and the first approximation to this motion, for z great, corresponds to a complex velocity and stream function w' consisting of a "source-term," namely,

$$w' = -S\log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

Now 2π times the strength of the source represented by the "source-term" must equal the rate at which liquid is being displaced by the intrusion of the internal boundary, and if the boundary is of the character shown in fig. 4 the rate of displacement is $2l\mathrm{V}$, where $2l$ is the breadth of the semi-infinite straight portion on the left.

Accordingly

$$l = \pi\mathrm{SV}^{-1}. \qquad (33)$$

If the boundary is as in fig. 3, $l = 0$ and so $\mathrm{S} = 0$; it is in any case clear that the motion of a finite rigid internal boundary cannot give rise to a "source-term" in the expansion of w' for z great.

Thus in order to secure that the transformation (29) shall correspond to a configuration like that of fig. 3, the constants must be arranged so as to make S zero.

17. *The Doublet-Term, and the Coefficient of Inertia.*—If S is zero the formula (30) takes the form

$$dz = \mathrm{V}^{-1}\{1+\mathrm{VD}w^{-2}\}\,dw,$$

which is equivalent, to the same degree of approximation, to

$$dw = (\mathrm{V}-\mathrm{D}z^{-2})\,dz,$$

and so to

$$w = \mathrm{V}z+\mathrm{D}z^{-1}. \qquad (34)$$

When by superposition the liquid is brought to "rest at infinity" and the internal boundary is moving to the right with velocity V, the first approximation to the motion at great distance consists of a "doublet-term," namely,

$$\phi'+i\psi' = w' = \mathrm{D}z^{-1}. \qquad (35)$$

Considering for a moment the translational motion, with velocity V parallel to the axis of x, of a finite solid or "ship" which is not necessarily symmetrical about a line in the direction of motion, it is seen that the full expression, in terms of polar co-ordinates, for the velocity-potential ϕ' of the surrounding liquid may be written in the form

$$\phi' = r^{-1}(\mathrm{D}\cos\theta+\mathrm{E}\sin\theta)+\Sigma a_m r^{-m}\cos(m\theta+\alpha_m) \quad (m = 2, 3, \ldots). \qquad (36)$$

Let Green's theorem be applied to the functions ϕ' and x in the region bounded internally by the solid moving boundary, the "ship," and externally by any curve. It then appears that

$$\int(\phi'\,\partial x/\partial n - x\,\partial\phi'/\partial n)\,ds,$$

where ds is the element of arc, and ∂n the element of outward drawn normal, has the same value for the internal solid boundary as for any surrounding curve. In particular the surrounding curve may be taken to be a circle whose radius tends to indefinite greatness, and for this it is seen that the terms of type $r^{-m}\cos(m\theta+\alpha_m)$ make no contribution to the value of the integral, so that the total value is $2\pi\mathrm{D}$.

At the inner boundary, if l be the x cosine of the outward normal, $\partial x/\partial n = l$, and $\partial\phi'/\partial n = -l\text{V}$; so the integral is equivalent to

$$\int l\,(x\text{V}+\phi')\,ds.$$

If A be the area of the section of the ship,

$$\int lx\,ds = \text{A},$$

and there results the equality

$$2\pi\text{D} = \text{VA}+\int l\phi'\,ds. \qquad (37)$$

If ρ be the density of the liquid, and ϖ the impulsive pressure required to generate the motion of the liquid instantaneously from rest, $\varpi = \rho\phi'$; consequently

$$\int l\phi'\,ds = \rho^{-1}\int l\varpi\,ds = \rho^{-1}\text{X}_0, \qquad (38)$$

where X_0 is the x component of the resultant impulsive pressure exerted by the ship on the surrounding liquid at the instant when the motion was set up. If the mass of the ship M be equal to that of the liquid which it displaces, as would normally be the case, $\text{A} = \rho^{-1}\text{M}$. Hence (37) is equivalent to

$$2\pi\text{D} = \rho^{-1}(\text{MV}+\text{X}_0). \qquad (39)$$

Now if (X, Y) be the impulse required to set up the whole motion, both the motion of the ship and the motion of the liquid, $\text{X} = \text{MV}+\text{X}_0$; consequently (39) may be written in the form

$$\text{D} = \text{X}/2\pi\rho. \qquad (40)$$

By using y instead of x in the application of GREEN's theorem it may be shown similarly that

$$\text{E} = \text{Y}/2\pi\rho.$$

Thus the components D, E of the doublet represented by the doublet-terms in the expansion of w' or ϕ' at great distance are proportional to the components of the total impulse which would generate the motion from rest.*

In the case of a ship which is symmetrical about the line of motion E and Y are zero, and the effective inertia of the system for longitudinal motion is XV^{-1} or $2\pi\rho\text{V}^{-1}\text{D}$.

* The corresponding theorem in three dimensions was given by the writer in a paper entitled "On Doublet Distributions in Potential Theory," 'Proc. Royal Irish Academy,' vol. XXXII, Sec. A, No. 4, 1914, § 14.

18. If the transformation (29) be reduced to the approximate form (30) it appears that

$$S = (p+q)\frac{(1-2\alpha)c-\lambda k}{\lambda+1}+(p-q)c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$VD = \frac{p+q-1}{2(p+q)}S^2+\frac{p-q}{p+q}S+\left\{\frac{2pq}{p+q}-2\alpha(1-\alpha)\frac{p+q}{\lambda+1}\right\}c^2. \quad . \quad . \quad . \quad (42)$$

Thus the condition for the absence of the source-term is

$$k = \left\{\frac{1-2\alpha}{\lambda}+\frac{p-q}{p+q}\left(1+\frac{1}{\lambda}\right)\right\}c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and the effective inertia of the ship (S being zero) is

$$2\pi\rho\left\{\frac{2pq}{p+q}-2\alpha(1-\alpha)\frac{p+q}{\lambda+1}\right\}\frac{c^2}{V^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The values of the parameters must of course be such that $c > k > -c$, and it is clear that there is a wide range of values of α, p, q, and λ which permit of this being true. It may be well, in connexion with (44), to recall the fact that c is of the dimensions of a length multiplied by a velocity, and that for a given configuration in the z plane c/V is independent of V.

Though the formula (44) does actually contain the law of the dependence of the inertia of the ship (having a certain type of shape) upon the angles $2p\pi$ and $2q\pi$ at bow and stern, it must not be assumed that the functional law corresponds to the explicit appearance of p and q in the formula. The other parameters are to be regarded not as data, but as functions of what ought properly to be the data, for example the length and breadth or the length and area of the ship. If λ and cV^{-1} were expressed in terms of such data the formula would show explicitly the required law, and would probably be a much more complicated function of p and q.

The specification of the shape of the ship can without difficulty be reduced to quadratures by putting w real on the right-hand side of formula (29), and equating the real and imaginary parts respectively to dx and $i\,dy$. This gives x and y as integrals of certain functions of a real parameter w, but the functions are generally so complicated as to make evaluation of the integrals difficult.

In the special case of $\alpha = \frac{1}{2}$, $q = p$, if dz be equivalent to $ds\exp(i\chi)$ for w real, it is seen that

$$\chi = -p\pi+2p\tan^{-1}\{(c^2-w^2)^{1/2}/\lambda w\},$$

$$ds = \{(\lambda^2-1)w^2+c^2\}^p(\lambda+1)^{-2p}(c^2-w^2)^{-p}V^{-1}dw,$$

so that, if the radius of curvature be R,

$$R = -\frac{ds}{d\chi} = \frac{\{(\lambda^2-1)w^2+c^2\}^{p+1}(c^2-w^2)^{1/2-p}}{2Vp\lambda c^2(\lambda+1)^{2p}},$$

$$= \frac{c\lambda^2\cos^{1-2p}(\chi/2p)}{2Vp(\lambda+1)^{2p}\{\lambda^2\cos^2(\chi/2p)+\sin^2(\chi/2p)\}^{3/2}}. \quad . \quad . \quad . \quad . \quad (45)$$

This formula shows that when $p < \frac{1}{2}$ the value of R for $w = \pm c$ is zero. It also shows that in the neighbourhood of the point where $\chi = 0$ the curve can be made as flat as may be desired by taking λ sufficiently small.

19. It is natural to expect that the parameter α of the transformation (29) will particularly affect the shape of the curve at the extremities of the linear range. In order to study the form of the curve in the immediate neighbourhood of the point $w = c$ it is convenient to put

$$-(x-x_0+iy)\exp(ip\pi) = \xi+i\eta,$$

where x_0 is the value of x for $w = c$; this makes ξ and η co-ordinates referred to the point $w = c$ as origin, the axis of ξ being tangential to the curve in the direction of w decreasing. In the expression on the right-hand side of the transformation w is put equal to $c-\epsilon$, and an approximation made on the supposition that ϵ is small. This leads to

$$d(\xi+i\eta) = \mathrm{K}\epsilon^{-p}+\mathrm{L}\epsilon^{\alpha-p}\exp(i\alpha\pi),$$

where

$$\mathrm{K} = \left\{\frac{\lambda(c-k)}{\lambda+1}\right\}^{p+q}\frac{1}{\mathrm{V}(2c)^q}, \qquad \mathrm{L} = \frac{p+q}{\lambda(c-k)}(2c)^{1-\alpha}\mathrm{K}.$$

Integration gives

$$\xi = \frac{\mathrm{K}}{1-p}\epsilon^{1-p}, \qquad \eta = \frac{\mathrm{L}}{1+\alpha-p}\sin\alpha\pi\,.\,\epsilon^{1+\alpha-p}, \qquad (46)$$

so that in the immediate neighbourhood of the corner

$$\eta \propto \xi^{1+\frac{\alpha}{1-p}}. \qquad (47)$$

The velocity of slip near the corner is $d\epsilon/d\xi$, and consequently is proportional to the $\{p/(1-p)\}^{\mathrm{th}}$ power of ξ.

20. *Motion of a Ship with Pointed Bow and Flat Stern.*—When the internal boundary in the z plane is symmetrical about a line in the direction of flow, and

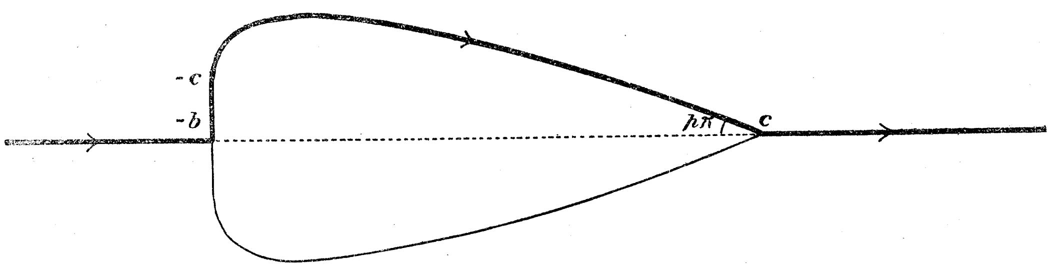

Fig. 5.

consists on either side of a curve departing from the line of symmetry at an angle $p\pi$ and rounding smoothly into a straight line which cuts the line of symmetry at right angles, the configuration has the general character of fig. 5.

The transformation must have angular range zero, and if $b > c$ a suitable type is

$$dz = \frac{C}{(w-c)^p (w+b)^{1/2}} \mathscr{C}^{p+1/2} dw, \qquad (48)$$

where $\mathscr{C}$ is a curve-factor of linear range c to $-c$, and angular range π. For example if the factor $\mathscr{C}_{10}$ be adopted the transformation is

$$dz = \frac{\{\lambda(w-k) + (w^2-c^2)^{1/2}\}^{p+1/2} dw}{V(\lambda+1)^{p+1/2}(w-c)^p (w+b)^{1/2}}. \qquad (49)$$

When this is expanded, for w great, in the form

$$dz = V^{-1}(1+Sw^{-1}+VDw^{-2})$$

it is found that

$$S = pc - \tfrac{1}{2}b - (p+\tfrac{1}{2})(\lambda+1)^{-1}k,$$

and

$$VD = \tfrac{1}{2}\frac{2p-1}{2p+1}S^2 + \frac{2}{2p+1}(pc-\tfrac{1}{2}b)S + \frac{p(b+c)^2}{2(2p+1)} - \tfrac{1}{4}(2p+1)c^2.$$

As S must vanish, the value of k must be

$$k = (\lambda+1)(pc-\tfrac{1}{2}b)/(p+\tfrac{1}{2}), \qquad (50)$$

and the other constants must be such that this k is between $-c$ and c.

The inertia-coefficient, for longitudinal motion, of the system consisting of the ship in liquid at rest at infinity, is

$$2\pi\rho V^{-2}\left\{\frac{p(b+c)^2}{2(2p+1)} - \tfrac{1}{4}(2p+1)c^2\right\}. \qquad (51)$$

Inflexional Curve-factors.

21. In seeking for curve-factors whose angular range shall be zero it will be observed that as w decreases through real values from $+c$ to $-c$ the vector angle of the complex

$$F(w) \equiv \lambda(w-k) + (w^2-c^2)^{1/2}, \qquad (52)$$

wherein $\lambda > 0$, will increase from zero through $\tfrac{1}{2}\pi$ to π, or will change from zero to $\pm\tan^{-1}\{c\lambda^{-1}(k^2-c^2)^{-1/2}\}$ as a maximum or minimum and then relapse to the value zero, according as $\lambda(w-k)$ does or does not change sign for a value of w in the linear range. Hitherto the angular range π has been secured by postulating $c^2 > k^2$; it may now be enquired whether, when $c^2 < k^2$, the function $F(w)$ can constitute or help to constitute a conformal curve-factor.

The product of $F(w)$ and its conjugate surd is a quadratic expression obviously possessed of real zeros, and so the only zeros for which $F(w)$ need be tested are real ones. If the case of $k > c$ be first considered, it is seen that $F(-\infty)$, $F(-c)$, $F(c)$ are negative, while $F(k)$ and $F(+\infty)$ are positive. Hence $F(w)$ has just one zero, namely, between c and k, the corresponding value g of w being

$$g = [\lambda^2 k - \{\lambda^2 k^2 - (\lambda^2 - 1) c^2\}^{1/2}]/(\lambda^2 - 1); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

the quantity g' derived from this by changing the sign of the square root is a zero of the expression conjugate to $F(w)$.

If $k < -c$, say $k = -k'$, it is seen that $F(-\infty)$ and $F(-k')$ are negative, while $F(-c)$, $F(c)$, and $F(+\infty)$ are positive. Thus $F(w)$ has just one zero, namely, between $-k'$ and $-c$, and its value $-h$ is given by

$$-h = [-\lambda^2 k' + \{\lambda^2 k'^2 - (\lambda^2 - 1) c^2\}^{1/2}]/(\lambda^2 - 1); \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

the quantity $-h'$ derived from this by changing the sign of the square root is a zero of the expression conjugate to $F(w)$.

In each of these cases the zero of $F(w)$ is of the first order. Hence if in the alternative cases attention be directed to $F(w)/(w-g)$ and $F(w)/(w+h)$ respectively, it is seen that these are functions free from zeros and infinities in the relevant region. Thus for $k > c$, $k' > c$, and $\lambda > 0$,

$$\mathscr{C}_{11} = \frac{\lambda(w-k) + (w^2 - c^2)^{1/2}}{w-g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$\mathscr{C}_{12} = \frac{\lambda(w+k') + (w^2 - c^2)^{1/2}}{w+h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

are conformal curve-factors. In each case the linear range is c to $-c$, the angular range is zero, and the order at infinity is zero. Within the linear range the modulus of $\mathscr{C}_{11}$ is $\surd\{(\lambda^2 - 1)(g' - w)/(g - w)\}$ and that of $\mathscr{C}_{12}$ is $\surd\{(\lambda^2 - 1)(w + h')/(w + h)\}$. It is to be noticed that, if $\lambda > 1$, $g' > k$ and $h' > k'$; if $\lambda < 1$, $g' < -c$ and $h' < -c$.

Any power of $\mathscr{C}_{11}$ or $\mathscr{C}_{12}$ is a curve-factor of inflexional character, and the greater the power the more pronounced is the variation in the direction of the tangent at different points of the curve. $\mathscr{C}_{11}$ corresponds to a curve which, as w decreases from c to $-c$, is first concave and afterwards convex to the relevant region; $\mathscr{C}_{12}$ corresponds to a curve which is first convex and afterwards concave. But any negative power of either corresponds to a curve in which, as compared with the original, the order of convexity and concavity is reversed.

22. *Hydrodynamical Example.*—The specification of a liquid stream disturbed by

the presence of a pear-shaped body of the general character shown in fig. 6 may be got by combining a Schwarzian factor, a curve-factor of angular range $\frac{1}{2}\pi$ for the

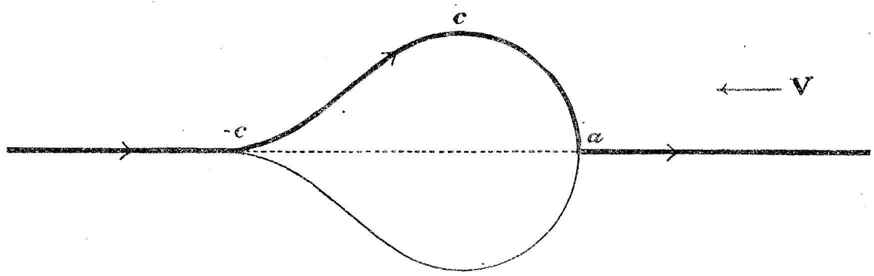

Fig. 6.

forward curve, and a power of $\mathscr{C}_{12}$ for the inflexional curve. Such a transformation would be

$$dz = \frac{[\mu(w-b)+\{(w-a)(w-c)\}^{1/2}]^{1/2}\mathscr{C}_{12}^{n}\,dw}{V(\mu+1)^{1/2}(\lambda+1)^{n}(w-a)^{1/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where $a > b > c$. The condition for the second straight stream-line being in the production of the first is

$$\tfrac{1}{2}a+n\left(\frac{k'}{\lambda-1}-h\right)-\tfrac{1}{4}\,\frac{2b+a+c}{\mu+1} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Curve-factors of Semi-infinite Linear Range.

23. For a range extending from $w = 0$ to $w = -\infty$ the simplest type of curve-factor is

$$\mathscr{C}_{13} = a^{1/2}+w^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where $a^{1/2} > 0$. Its angular range is $\frac{1}{2}\pi$ and its order at infinity is $\frac{1}{2}$. The transformation

$$dz = \frac{\mathscr{C}_{13}^{2p}\,dw}{Vw^{p}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

gives a boundary consisting of a straight line and a curve intersecting it at an angle $p\pi$. In the special case of $p = \frac{1}{2}$ the curve is a semi-parabola.

It may be noted that when w is negative $|\mathscr{C}_{13}| = (a-w)^{1/2}$.

24. If a more general expression be considered, namely

$$\mathscr{C} = w-k+(cw)^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where $c > 0$, it is found that a distinction has to be drawn between the case when $k > 0$ and the case when $k < 0$.

Let $k > 0$, and let $\mathscr{D}$ be the surd conjugate to $\mathscr{C}$. Then

$$\mathscr{C}\mathscr{D} = (w-k)^2 - cw,$$

which is positive for $w = 0$, negative for $w = k$, and positive for $w = +\infty$. Hence $\mathscr{C}\mathscr{D}$ has real roots, of which one (g) is between 0 and k, and the other (g') is greater than k. The values are

$$g = k+\tfrac{1}{2}c-(kc+\tfrac{1}{4}c^2)^{1/2}, \quad g' = k+\tfrac{1}{2}c+(kc+\tfrac{1}{4}c^2)^{1/2}. \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Now $\mathscr{C}$ is negative for $w = 0$, positive for $w = k$ and for $w = +\infty$. Hence $\mathscr{C}$ has one zero, which must be g. But

$$\mathscr{C}_{14} = \frac{w-k+(cw)^{1/2}}{w-g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

wherein $k > 0$, has no zeros and is a conformal curve-factor. Its order at infinity is zero, and therefore its angular range ought to be zero. This is readily verified, for if χ be the vector angle of $\mathscr{C}_{14}$ when w is real and negative

$$\begin{aligned} -\tan\chi &= (-cw)^{1/2}/(-w+k) \\ &= c^{1/2}/\{(-w)^{1/2}+k\,(-w)^{-1/2}\}. \end{aligned}$$

As $-w$ increases from zero the denominator of this expression diminishes from $+\infty$ to a minimum value corresponding to $-w = k$, namely $2k^{1/2}$, and then increases again to infinity. Hence χ diminishes from zero to a minimum value $-\tan^{-1}(c^{1/2}/2k^{1/2})$, and then increases again to zero.

When w is negative the modulus of $\mathscr{C}_{14}$ is $\{(g'-w)/(g-w)\}^{1/2}$.

When $k < 0$ let it equal $-k'$; then $\mathscr{C}$ takes the form

$$\mathscr{C}_{15} = w+k'+(cw)^{1/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

This can have no real zeros, since the two terms when both real are also both positive. There can be no imaginary zeros in the relevant region because the vector angle is $\chi = \tan^{-1}\{(-cw)^{1/2}/(w+k')\}$, which increases from zero (for $w = 0$) through $\frac{1}{2}\pi$ (for $w = -k'$) to π (for $w = -\infty$) and so gives an angular range π, while the order at infinity is unity. Thus $\mathscr{C}_{15}$ is a conformal curve-factor.

When w is negative

$$|\mathscr{C}_{15}|^2 = \{w+(k'-\tfrac{1}{2}c)\}^2 - \{\tfrac{1}{4}c^2 - ck'\},$$

which has real or imaginary factors according as c is greater or less than $4k'$. In the former case c may be put equal to $4k'\cosh^2\beta$, so that $\mathscr{C}_{15}$ takes the form

$$\mathscr{C}_{16} = w+k'+2\,(k'w)^{1/2}\cosh\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

and it appears that when w is negative

$$|\mathscr{C}_{16}|^2 = (k'e^{2\beta}-w)(k'e^{-2\beta}-w). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

The utility of a knowledge of the moduli of curve-factors will be exemplified later.

24*a*. *Curve-Factors involving Powers other than the Square Root.*—A more general type of curve-factor having semi-infinite range is obtainable as follows. Consider

$$\mathscr{C}_{17} = b^{\alpha}+w^{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where $\frac{1}{2} > \alpha > 0$, and both b and b^{α} are positive. This has no real zeros, since both terms are positive when $w > 0$. The linear range of curvilinearity is from zero to $-\infty$, and on this range w^{α} is represented by $(-w)^{\alpha} \exp(i\alpha\pi)$, so that the vector angle χ of $\mathscr{C}_{17}$ is

$$\chi = \tan^{-1}\left\{\frac{(-w)^{\alpha}\sin\alpha\pi}{b^{\alpha}+(-w)^{\alpha}\cos\alpha\pi}\right\}.$$

The denominator in this expression cannot vanish, so, as w decreases from zero, χ increases to $\alpha\pi$; thus the angular range is $\alpha\pi$. The order at infinity is α, so there are no imaginary zeros in the relevant region. Hence $\mathscr{C}_{17}$ is a curve-factor.

The case of $1 > \alpha > \frac{1}{2}$ has been excluded because it would give an angular range π, and so introduce imaginary zeros into $\mathscr{C}_{17}$.

25. Consider also

$$\mathscr{C}_{18} = w+k'+b^{1-\alpha}w^{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

where $k' > 0$, $\frac{1}{2} > \alpha > 0$, and both b and $b^{1-\alpha}$ are positive. This has no real zeros since all the terms are positive when $w > 0$. On the range of curvilinearity the vector angle χ is

$$\chi = \tan^{-1}\left\{\frac{b^{1-\alpha}(-w)^{\alpha}\sin\alpha\pi}{w+k'+b^{1-\alpha}(-w)^{\alpha}\cos\alpha\pi}\right\}.$$

The denominator in this expression is negative for $w=-\infty$ and positive for $w=-k'$ and $w=0$, and so has a zero for a value less than $-k'$. Therefore as w decreases from zero χ increases through $\frac{1}{2}\pi$ to π, and the angular range is π. The order at infinity is unity, so there are no imaginary zeros in the relevant region. Hence $\mathscr{C}_{18}$ is a conformal curve-factor.

26. The function

$$\mathscr{C}_{19} = \lambda+\mu\{(w+a)w\}^{1/2}+\nu w^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

wherein λ, μ, ν, and a are all positive, is real for w real and positive. For $0 > w > -a$ it has the vector angle

$$\tan^{-1}[\mu\{(w+a)(-w)\}^{1/2}+\nu(-w)^{1/2}]/\lambda,$$

and for $w < -a$ the vector angle

$$\tan^{-1}[\nu(-w)^{1/2}/\{\lambda-\mu(a-w)^{1/2}(-w)^{1/2}\}],$$

so that it has an angular range π corresponding to the linear range 0 to $-\infty$. The order at infinity being unity there are no imaginary zeros in the relevant region, and so $\mathscr{C}_{19}$ is a conformal curve-factor. It has something of the character of a double curve-factor.

Double Curve-factors.

27. An example of a double curve-factor is afforded by the function

$$\mathscr{C}_{20} = w^2-\tfrac{1}{2}(a^2+b^2)+\{(w^2-a^2)(w^2-b^2)\}^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where $b > a > 0$. The form of this must be considered for different parts of the axis of w real.

For $w > b$ the function is real; moreover $w^2-\frac{1}{2}(a^2+b^2)$ is positive, so $\mathscr{C}_{20}$ has no zero in this part of the axis of w real.

For $b > w > a$ the form is

$$w^2-\tfrac{1}{2}(a^2+b^2)+i(w^2-a^2)^{1/2}(b^2-w^2)^{1/2},$$

of which the real part changes from positive to negative as w diminishes through the value $+\{\frac{1}{2}(a^2+b^2)\}^{1/2}$. The vector angle accordingly ranges from zero to π.

For $a > w > -a$ the form is

$$-\{\tfrac{1}{2}(a^2+b^2)-w^2\}-(a^2-w^2)^{1/2}(b^2-w^2)^{1/2},$$

of which both terms are real and negative.

For $-a > w > -b$ the form is

$$-\{\tfrac{1}{2}(a^2+b^2)-w^2\}-i(w^2-a^2)^{1/2}(b^2-w^2)^{1/2},$$

of which the first term changes from negative to positive as w diminishes through the value $-\{\frac{1}{2}(a^2+b^2)\}^{1/2}$. The vector angle increases from π to 2π.

For $w < -b$ the form is

$$w^2-\tfrac{1}{2}(a^2+b^2)+(w^2-a^2)^{1/2}(w^2-b^2)^{1/2},$$

of which both terms are positive.

Thus the factor corresponds to two curves separated by a straight line, and over the whole range from $+b$ to $-b$ the angular amplitude is 2π. The order at infinity is 2, so there are no imaginary zeros in the relevant region. It appears therefore that $\mathscr{C}_{20}$ is a curve-factor.

28. It is interesting to enquire what sort of a configuration is given by a transformation which involves $\mathscr{C}_{20}$ and also Schwarzian factors introducing corners at the points $\pm a$, $\pm b$. Such a transformation is

$$dz = \frac{A\mathscr{C}_{20}^{n}\,dw}{(w-a)^p(w+a)^{p'}(w-b)^q(w+b)^{q'}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and it gives a configuration of the character indicated in fig. 7, the second and third straight lines being inclined to the first at angles $(n-p-q)\,\pi$ and $(2n-p-q-p'-q')\,\pi$ respectively.

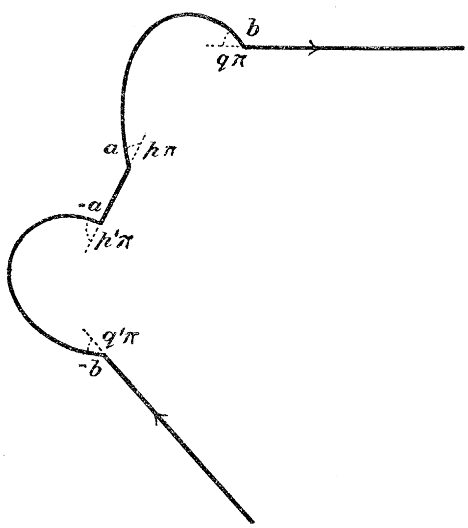

Fig. 7.

In order that the three lines should be parallel it would be necessary to have

$$n = p+q = p'+q'. \quad . \quad . \quad . \quad (72)$$

If it were further desired that the first and last lines should be parts of the same straight line the constants would have to be so adjusted as to make the term in w^{-1} vanish in the expansion of dz/dw for w great. This condition is equivalent to

$$a\,(p-p')+b\,(q-q') = 0,$$

which, when combined with (72), gives

$$p' = p = n-q = n-q'. \quad . \quad . \quad (73)$$

Hence the transformation

$$dz = \frac{1}{2^n V}\,\frac{\mathscr{C}_{20}^n\,dw}{(w^2-a^2)^{n-q}\,(w^2-b^2)^q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

gives a configuration of the character of the thick line in fig. 8, which, in the hydrodynamical application, may be duplicated by reflexion.

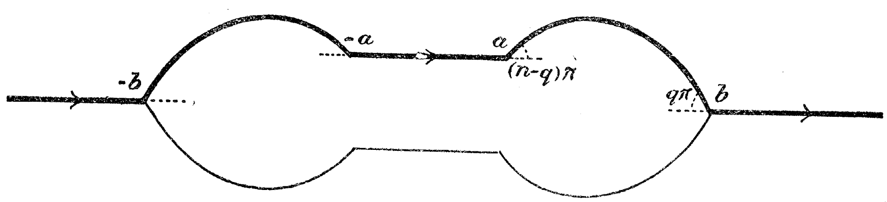

Fig. 8.

The particular case of $q = \frac{1}{2}$ gives a dumb-bell shaped boundary as in fig. 9, and the case of $n = q$ gives a ship with straight sides and pointed ends, as in fig. 10.

29. A double curve-factor containing a greater number of arbitrary parameters than $\mathscr{C}_{20}$ is

$$\mathscr{C}_{21} = \lambda\,(w-g)\,(w-h)+\{(w-a)\,(w-b)\,(w-c)\,(w-d)\}^{1/2}, \quad . \quad . \quad . \quad (75)$$

where λ is positive, and a, g, b, c, h, d are real constants in descending order of magnitude. The use of this factor instead of $\mathscr{C}_{20}$ in the applications of the previous article would give greater variety in the possible shapes of boundary. The constants could probably be adjusted so as to make all three straight portions of the boundary parts of one line, and so lead to a specification of liquid flow past two objects of oval form.

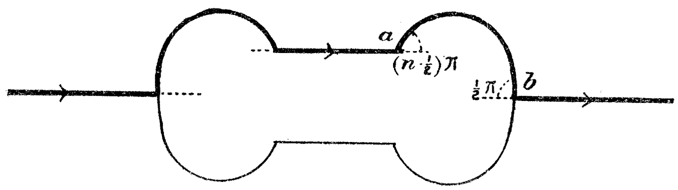

Fig. 9.

Another double curve-factor, corresponding to two curves separated by a straight line, is

$$\mathscr{C}_{22} = \lambda\,(w-a)^{1/2}\,(w-c)^{1/2} + (w-b)^{1/2}\,(w-d)^{1/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

where $\lambda > 0$, and $a > b > c > d$. It has a total angular range of π, made up of $\frac{1}{2}\pi$ from each of the two curved portions.

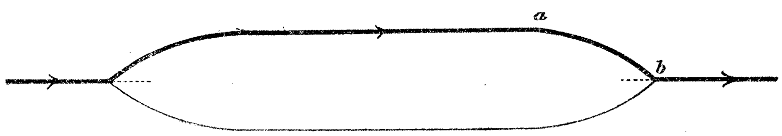

Fig. 10.

It would be interesting to know whether a double curve-factor is necessarily the product or other simple function of simple curve-factors.

Synthesis of Curve-factors from Elementary Types.

30. A simple function which has not yet been considered as a possible curve-factor is

$$\mathrm{F}\,(w) = k-w+(w^2-c^2)^{1/2}.$$

The product of this and its conjugate surd has only one zero, a real one, so $\mathrm{F}\,(w)$ need only be tested for real zeros.

It is to be noticed that

$$\frac{d}{dw}\,\{w-(w^2-c^2)^{1/2}\} = 1-\frac{w}{(w^2-c^2)^{1/2}},$$

which is negative both for w real, positive and greater than c, and for w real, negative and less than $-c$, due regard being paid to the significance of the square root. Thus $\{w-(w^2-c^2)^{1/2}\}$ is positive for w real and greater than c, and continually decreases as w increases, having the limit zero for $w \to +\infty$; for w real and less than $-c$, $\{w-(w^2-c^2)^{1/2}\}$ is negative and continually increases to the limit value zero as $w \to -\infty$. It follows that $\mathrm{F}\,(w)$ has no stationary values for w real and $w^2 > c^2$; the values of $\mathrm{F}\,(w)$ for $w = -\infty$, $w = -c$, $w = c$, and $w = +\infty$, are k, $k+c$, $k-c$, and

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k respectively. The signs of these four values of $F(w)$ are all positive if $k > c$, and all negative if $k < -c$. But if $c > k > 0$ the signs are $+, +, -, +$, and if $0 > k > -c$ the signs are $-, +, -, -$.

Thus, if $k^2 < c^2$, $F(w)$ changes sign either between $w = c$ and $w = +\infty$ or between $w = -c$ and $w = -\infty$; accordingly $F(w)$ has a zero and is not a curve-factor. But, if $k^2 > c^2$, $F(w)$ does not change sign on either of the above-named ranges of w, and so has no zeros. Thus for $k^2 > c^2$ there is a curve-factor

$$\mathscr{C}_{23} = k - w + (w^2 - c^2)^{1/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

having the linear range $+c$ to $-c$, the angular range zero, and the order at infinity zero.

31. On examination it will be found that some of the curve-factors already discussed are resolvable into factors of simpler type. If in $\mathscr{C}_5$ the parameter k be replaced by the introduction of β, such that

$$k \sinh \alpha = c \sinh \beta, \qquad (\alpha > \beta > -\alpha),$$

it is readily verified that

$$\mathscr{C}_5 \equiv w \sinh \alpha - c \sinh \beta + (w^2 - c^2)^{1/2} \cosh \alpha$$
$$= (2c)^{-1} e^{\beta} \{w + ce^{-(\alpha+\beta)} + (w^2 - c^2)^{1/2}\} \{ce^{\alpha-\beta} - w + (w^2 - c^2)^{1/2}\}, \quad . \quad . \quad (78)$$

so that $\mathscr{C}_5$ is equivalent to the product of two factors, one of the type of $\mathscr{C}_6$, the other of the type of $\mathscr{C}_{23}$.

Similarly if in $\mathscr{C}_7$ a parameter β be introduced such that

$$k \cosh \alpha = \pm c \cosh \beta, \qquad (\alpha > \beta > 0),$$

it appears that

$$\mathscr{C}_7 \equiv w \cosh \alpha - (\pm) c \cosh \beta + (w^2 - c^2)^{1/2} \sinh \alpha$$
$$= \pm (2c)^{-1} e^{\beta} \{w - (\pm) ce^{-(\alpha+\beta)} + (w^2 - c^2)^{1/2}\} \{(\pm) ce^{\alpha-\beta} - w + (w^2 - c^2)^{1/2}\}, \quad . \quad (79)$$

so that $\mathscr{C}_7$ is equivalent to the product of two factors, one of the type of $\mathscr{C}_6$, the other of the type of $\mathscr{C}_{23}$.

$\mathscr{C}_{11}$ and $\mathscr{C}_{12}$, reclassified according as λ is greater or less than unity, are equivalent to

$$\mathscr{C}_{24} = \frac{w \cosh \alpha \mp c \cosh \beta + (w^2 - c^2)^{1/2} \sinh \alpha}{w \mp c \cosh(\beta - \alpha)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

in which $\beta > \alpha$, and

$$\mathscr{C}_{25} = \frac{w \sinh \alpha - c \sinh \beta + (w^2 - c^2)^{1/2} \cosh \alpha}{w - c \cosh(\beta - \alpha)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

in which $\beta^2 > \alpha^2$. It can be readily verified that

$$\mathscr{C}_{24} = e^{\alpha} \frac{w \mp ce^{-(\alpha+\beta)} + (w^2-c^2)^{1/2}}{w \mp ce^{-(\beta-\alpha)} + (w^2-c^2)^{1/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

$$\mathscr{C}_{25} = e^{\alpha} \frac{w + ce^{-(\alpha+\beta)} + (w^2-c^2)^{1/2}}{w - ce^{-(\beta-\alpha)} + (w^2-c^2)^{1/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Consequently both these types of inflexional curve-factors are equivalent to simple fractions having curve-factors of the type of $\mathscr{C}_6$ in numerator and denominator.

In the case of a semi-infinite linear range it is seen that

$$\mathscr{C}_{16} \equiv w + k' + 2(k'w)^{1/2} \cosh \beta = (k'^{1/2}e^{-\beta} + w^{1/2})(k'^{1/2}e^{\beta} + w^{1/2}), \quad . \quad . \quad . \quad (84)$$

so that $\mathscr{C}_{16}$ is equivalent to the product of two curve-factors of the type of $\mathscr{C}_{13}$. Also (with the notation of article 24)

$$\mathscr{C}_{14} \equiv \frac{w-k+(cw)^{1/2}}{w-g} = \frac{g'^{1/2}+w^{1/2}}{g^{1/2}+w^{1/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

in which fraction both numerator and denominator are of the type of $\mathscr{C}_{13}$.

32. Of the curve-factors which have been studied some are such that, for real values of w which render them complex, the squares of their moduli have real factors; others have not this property. To the latter class the theorems of the previous article do not apply, but as regards the former class the results suggest a useful method of generalisation. Within the limits of this class of curve-factors it seems possible to regard the types $\mathscr{C}_6$ and $\mathscr{C}_{23}$ as fundamental for the finite linear range $-c$ to $+c$, and the type $\mathscr{C}_{13}$ for the semi-infinite range $-\infty$ to zero. Bearing in mind that any power of a curve-factor, whether positive or negative, is itself a curve-factor, one may take any number of curve-factors of the types $\mathscr{C}_6$ and $\mathscr{C}_{23}$, raise each to any power, and multiply all together, so producing a resultant curve-factor for the assigned finite linear range. In the same way one may take the product of arbitrary powers of any number of curve-factors of the type $\mathscr{C}_{13}$, so getting a resultant curve-factor for the assigned semi-infinite range. These curve-factors have a quite arbitrary number of adjustable parameters, and so correspond to an endless multiplicity of curves in the plane of z.

If, for example, the ship problem already discussed in article 15 be again considered, it is legitimate to take a set of parameters $k_1, k_2, \ldots k_r, \ldots$ intermediate in value between $-c$ and $+c$, and a set of corresponding indices $n_1, n_2, \ldots n_r, \ldots$; also a set of parameters $l_1, l_2, \ldots l_s, \ldots$ greater than c or less than $-c$, and a set of corresponding indices $m_1, m_2, \ldots m_s, \ldots$. These combine to give the curve-factor

$$\mathscr{C}_{26} = \Pi \{w-k_r+(w^2-c^2)^{1/2}\}^{n_r} \Pi \{l_s-w+(w^2-c^2)^{1/2}\}^{m_s}, \quad . \quad . \quad . \quad . \quad (86)$$

leading to the transformation

$$dz = \frac{A\mathscr{C}_{26}\, dw}{(w-c)^p\,(w+c)^q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (87)$$

The parameters must comply with the condition for zero angular range, namely

$$\Sigma n_r = p+q,$$

and with the condition for the vanishing of the source-term in w. Subject to these, the constants are arbitrary and are in theory adjustable to meet a corresponding number of possible requirements.

The range of adjustable parameters in $\mathscr{C}_{26}$ seems, however, to be unexpectedly and embarrassingly great, and there suggests itself a doubt whether the degree of generality of the formula can be really so high as it appears to be. It is noticeable that the m_s parameters do not enter into the condition for zero angular range, and it seems worth while to study the $\mathscr{C}_{23}$ type of curve-factor more closely.

When w is real and $c > w > -c$,

$$|\mathscr{C}_{23}|^2 = k^2+c^2-2kw,$$

which is not formally a product or quotient of rational integral functions of w. So far as this is true, $\mathscr{C}_{23}$ could not be expressed as the product or quotient of curve-factors the squares of whose moduli are rational integral functions of w. But it must not be forgotten that there is a curve-factor, namely $\mathscr{C}_1 = w+(w^2-c^2)^{1/2}$, whose modulus is a constant. The use of this gives a fractionalisation of $\mathscr{C}_{23}$, for it is readily verified that

$$\mathscr{C}_{23} = k-w+(w^2-c^2)^{1/2} = k\,\frac{w-c^2k^{-1}+(w^2-c^2)^{1/2}}{w+(w^2-c^2)^{1/2}},$$

wherein, as $k^2 > c^2$, $(c^2k^{-1})^2 < c^2$, so that $\mathscr{C}_{23}$ is a fraction whose numerator is of the type of $\mathscr{C}_6$ and whose denominator is $\mathscr{C}_1$. Moreover, it is to be noticed that $\mathscr{C}_1$ is only a particular case of $\mathscr{C}_6$.

There is therefore no loss of generality in discarding $\mathscr{C}_{23}$ from the category of fundamental types, and in forming new curve-factors by combining arbitrary powers of different forms of $\mathscr{C}_6$ only.

Thus the curve-factor

$$\mathscr{C}_{27} = \Pi\,\{w-k_r+(w^2-c^2)^{1/2}\}^{n_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (88)$$

is no less general than $\mathscr{C}_{26}$. It may be substituted for $\mathscr{C}_{26}$ in the transformation (87) for the problem of the doubly pointed ship, with the same condition for zero angular range.

33. The formula just obtained obviously suggests the further step of establishing a functional relation between n_r and k_r, and letting k_r range over all real values subject to the limitation $c > k_r > -c$.

The process would be to take a real variable θ representing k_r and to put $f(\theta)\,d\theta$ for n_r, $f(\theta)$ being a real function not involving w. It would generally be legitimate to replace the limit of the product in formula (88) by the exponential of the limit of the sum of the logarithms of the various factors, which last limit is an integral. There results the very general curve-factor

$$\mathscr{C}_{28} = \exp \int_{-c}^{c} f(\theta) \log \{w-\theta+(w^2-c^2)^{1/2}\}\, d\theta, \quad . \quad . \quad . \quad . \quad . \quad (89)$$

in which there is no *à priori* restriction of $f(\theta)$ to continuity, beyond the exclusion of such infinities as would prevent convergence of the formula. There may, for example, be such infinities, for particular values of θ, as would make $\mathscr{C}_{28}$ include as factors definite powers of the corresponding forms of $\mathscr{C}_6$.

For values of w on the linear range the modulus of $\mathscr{C}_{28}$ is

$$\exp \int_{-c}^{c} \tfrac{1}{2} f(\theta) \log (\theta^2-2\theta w+c^2)\, d\theta,$$

and the vector angle

$$\int_{-c}^{c} f(\theta) \tan^{-1} \{(c^2-w^2)^{1/2}/(w-\theta)\} d\theta.$$

The angular range is

$$\pi \int_{-c}^{c} f(\theta)\, d\theta.$$

Liquid Motions with Free Stream-lines.

34. In a liquid motion the characteristic of a stream-line which is free, or is a possible line of discontinuity in the motion, is that along it the resultant velocity q is constant. Now $q^{-1} = |dz/dw|$, and therefore a transformation of the type $dz/dw = f(w)$ will give as part of the boundary a possible free stream-line if there is a part of the range w real for which $|f(w)|$ is constant.

The simplest example of this is presented by the curve-factor $\mathscr{C}_1$; since the modulus of this, within its linear range, is constant, a transformation in which $f(w)$ consists solely of a power of $\mathscr{C}_1$ gives a free stream-line as part of the boundary. In fact, the transformation

$$dz = \mathrm{A}\, \{w+(w^2-c^2)^{1/2}\}^n\, dw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

gives a configuration of the general character indicated by fig. 1, with this special feature that on the curved part of the boundary there indicated the velocity is constant. Thus, so far as the ordinary theory of discontinuous fluid motion is concerned, the curved part of the boundary may be a free stream-line.

35. Other cases of free stream-lines across finite gaps in boundaries which are otherwise rectilineal may be built up from curve-factors of the types $\mathscr{C}_6$ and $\mathscr{C}_{23}$. These being taken in the forms

$$\mathscr{C}_6 = w-ce^{-\gamma}+(w^2-c^2)^{1/2},$$

$$\mathscr{C}_{23} = ce^{\gamma}-w+(w^2-c^2)^{1/2},$$

it is seen that their respective moduli (on the linear range) are

$$\{2ce^{-\gamma}(c\cosh\gamma - w)\}^{1/2} \quad \text{and} \quad \{2ce^{\gamma}(c\cosh\gamma - w)\}^{1/2},$$

and it is to be remembered that the angular range of $\mathscr{C}_{23}$ is zero. Thus the transformation

$$dz = \frac{e^{-(p-m)\gamma}\,dw}{\mathrm{U}\,(2c)^p\,(w - c\cosh\gamma)^p}\,\mathscr{C}_6^m\,\mathscr{C}_{23}^{2p-m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)^*$$

representing a configuration of the kind indicated in fig. 11, gives $|dz/dw| = \mathrm{U}^{-1}$ along the curved part of the boundary, so that the curve may be a free stream-line along which the velocity is U. If m be taken equal to p the angular range of the whole transformation is zero, so that the first and last straight stream-lines are parallel.

The extension to boundaries with a greater number of corners is obvious.

36. Another simple example, leading to well-known results, is afforded by the use of $\mathscr{C}_{13}$. As the modulus of $a^{1/2} + w^{1/2}$ for w negative is $(a-w)^{1/2}$, the combination of

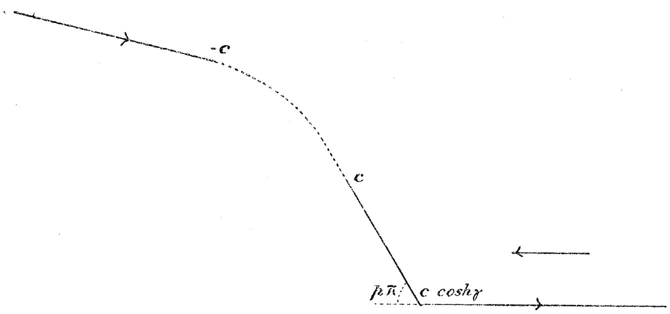

Fig. 11.

a power of $\mathscr{C}_{13}$ with a Schwarzian factor consisting of a suitable power of $w-a$ will give a free stream-line.

In fact, the transformation

$$dz = \frac{(a^{1/2} + w^{1/2})^{2p}\,dw}{\mathrm{V}\,(w-a)^p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

gives a liquid flow of the character indicated in fig. 12, there being a fixed obstacle consisting of two planes meeting at an angle $2p\pi$, and stream-lines extending to infinity and tending to parallelism with the undisturbed stream.

The case of $p = \frac{1}{2}$ is that of an obstacle consisting of a single-plane wall at right angles to the stream.

37. The previous transformation suggests a method of building up a transformation applicable to symmetrical flow past an obstacle in the form of an open polygon. The method consists in employing for the range of w negative, (the range which is to

* An equivalent form of this transformation is

$$dz = \frac{e^{p\gamma}c^{p-m}\,dw}{2^p\mathrm{U}\,(w - c\cosh\gamma)^p}\,\mathscr{C}_6^{2p}\,\mathscr{C}_1^{m-2p}.$$

correspond to a free stream-line), the product of powers of a number of curve-factors of the type of $\mathscr{C}_{13}$, with different parameters.

Suppose the configuration of liquid flow, (halved by the line of symmetry), to be of

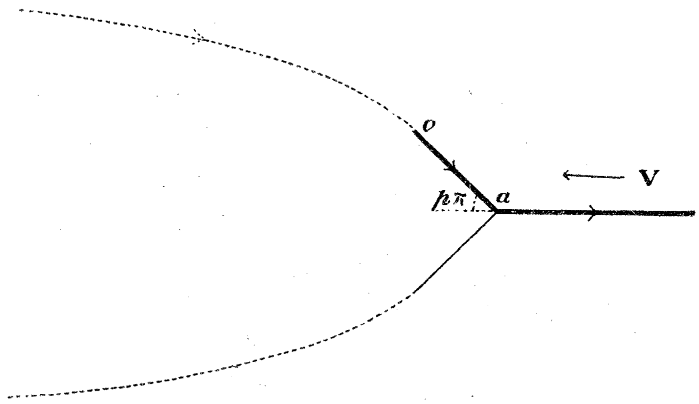

Fig. 12.

the character indicated in fig. 13, the fixed obstacle being polygonal with corners A, B, C, and corresponding external angles $p\pi$, $q\pi$, $r\pi$. Let values a, b, c be assigned to w at A, B, C, these being in descending order of magnitude, and let the value of w at the end D of the polygon be zero. Consider the transformation

$$dz = \frac{(a^{1/2}+w^{1/2})^{2p}\,(b^{1/2}+w^{1/2})^{2q}\,(c^{1/2}+w^{1/2})^{2r}}{\mathrm{V}\,(w-a)^p\,(w-b)^q\,(w-c)^r}\,dw\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

this gives all the corners as specified, and has a constant modulus for w negative. Hence the curve DE is a free stream-line.

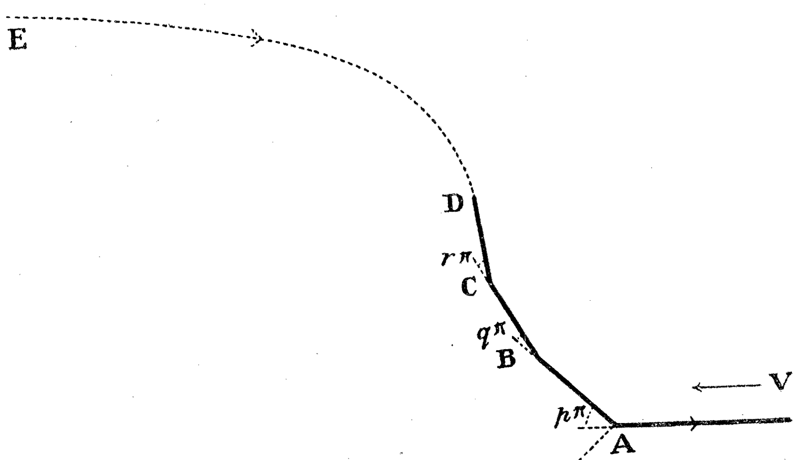

Fig. 13.

The parameters are adjustable so as to make the dimensions of the polygon such as may be desired.

It need hardly be said that a reëntrant angle of the polygon would, through the corresponding Schwarzian factor, lead to a zero of dz/dw, and so to an infinite velocity. In such a case some different transformation would apply, namely one which would include a free stream-line starting from the reëntrant corner, at least one of the subsequent corners being in the region of still water.

38. *Free Stream-lines when the Fixed Boundary includes Curves.*—The method of the previous article suggests a step towards generalisation which consists in taking part of the polygonal fixed boundary to have an indefinitely large number of corners each of indefinitely small external angle, introducing a suitable curve-factor and Schwarzian factor for each corner, and taking the limit of the product.

The parameters $b, c, \ldots$ are replaced by a real variable θ which ranges between the values assigned to w at the extremities of the curved part of the fixed boundary, and the indices $q, r, \ldots$ of the corresponding Schwarzian factors are represented by $d\chi/\pi$, where χ is the angle between the tangent, drawn in the sense of w increasing, and a fixed direction.

Thus there is obtained the resultant curve-factor

$$\mathscr{C} = \text{Lim}\ \Pi\ \{(w^{1/2}+\theta^{1/2})^2\,(w-\theta)^{-1}\}^{d\chi/\pi},$$

which is generally equivalent to

$$\mathscr{C}_{29} = \exp \frac{1}{\pi}\int \log\left(\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right)d\chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

the integration being extended over a suitable range of real values of θ.

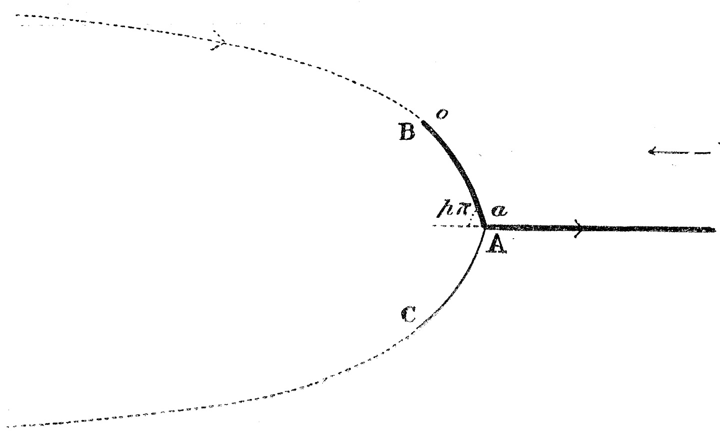

Fig. 14.

For example if a liquid flow be interrupted by a symmetrical and symmetrically placed obstacle consisting of finite curves CA, AB meeting at A at an angle $2p\pi$, and the values of w at B and A be taken as zero and a respectively, the transformation would be of the form

$$dz = \mathrm{K}\,\mathscr{C}_{30}\frac{(w^{1/2}+a^{1/2})^{2p}}{(w-a)^p}\,dw, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

where K is a real constant, and

$$\mathscr{C}_{30} = \exp \frac{1}{\pi}\int_{\theta=o}^{\theta=a} \log\left(\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right) d\chi. \quad (96)$$

Clearly the formula (96) does not become definite until a functional relation is specified between χ and θ. There are two ways in which the attempt may be made to utilise the formula. One is the purely empirical device of assigning arbitrarily such relations between χ and θ as would make the definite integral susceptible of precise evaluation, and then ascertaining the corresponding configurations of fixed and free boundaries. The other is to seek the particular functional relation which shall correspond to an arbitrarily assigned form of fixed boundary.

If $\chi = f(\theta)$, $\mathscr{C}_{30}$ takes the form

$$\mathscr{C}_{30} = \exp \frac{1}{\pi}\int_{o}^{a} f'(\theta) \log\left(\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right) d\theta, \quad (97)$$

wherein it may be noted that the logarithmic infinity, which would correspond to $\theta = w$ in the particular case of w real, does not destroy the convergence of the integral. On substitution of this in (95), dz being represented by $ds \exp(i\chi)$, it appears that when w is real and between zero and a,

$$\chi = -p\pi - \int_{w}^{a} f'(\theta)\, d\theta = f(w),$$

$f(a)$ being taken equal to $-p\pi$, and

$$ds = \mathrm{K}\, dw \left(\frac{w^{1/2}+a^{1/2}}{a^{1/2}-w^{1/2}}\right)^{p} \exp \frac{1}{\pi}\int_{o}^{a} f'(\theta) \log\left|\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right| d\theta.$$

Suppose the intrinsic equation to the curved part of the fixed boundary to be prescribed, so that

$$ds/d\chi = \mathrm{F}(\chi) = \mathrm{F}\{f(w)\},$$

where F is a prescribed function. Then it appears that

$$\mathrm{F}\{f(w)\} = \frac{\mathrm{K}}{f'(w)}\left(\frac{a^{1/2}+w^{1/2}}{a^{1/2}-w^{1/2}}\right)^{p} \exp \frac{1}{\pi}\int_{o}^{a} f'(\theta) \log\left|\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right| d\theta. \quad (98)$$

This relation specifies the property which the unknown function $f(w)$ is required to satisfy. It is an integral equation in $f'(w)$, of by no means encouraging appearance.

39. It is clear that the practically useful method of applying formula (97) is to assign a convenient form to $f(\theta)$ and to enquire what sort of problems can thereby be solved. To this end it is convenient to separate $\mathscr{C}_{29}$ or $\mathscr{C}_{30}$ in the first instance from any transformation such as (95) and to study it as a double curve-factor having two contiguous linear ranges, namely from $-\infty$ to 0, and from 0 to a, with the special property that on the former of these ranges it has its modulus independent of w.

It is convenient to remove the logarithm from the integral by integration by parts, thus

$$\pi \log \mathscr{C}_{30} = \int_o^a f'(\theta) \log\left(\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right) d\theta$$

$$= \left[f(\theta) \log\left(\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right)\right]_o^a - \int_o^a f(\theta)\left(\frac{w}{\theta}\right)^{1/2} \frac{d\theta}{w-\theta}. \quad . \quad . \quad . \quad (99)$$

By way of illustration, let $f(\theta) = \lambda\theta^{1/2}$, then indicating the particular case by a fresh suffix,

$$\pi \log \mathscr{C}_{31} = \left[\lambda\theta^{1/2} \log\left(\frac{w^{1/2}+\theta^{1/2}}{w^{1/2}-\theta^{1/2}}\right) + \lambda w^{1/2} \log (w-\theta)\right]_o^a,$$

and

$$\mathscr{C}_{31} = \left(\frac{w^{1/2}+a^{1/2}}{w^{1/2}-a^{1/2}}\right)^{\frac{\lambda a^{1/2}}{\pi}} \left(\frac{w-a}{w}\right)^{\frac{\lambda w^{1/2}}{\pi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

For w real and greater than a, $\mathscr{C}_{31}$ is clearly real. For $a > w > 0$ it takes the form

$$\left(\frac{w^{1/2}+a^{1/2}}{a^{1/2}-w^{1/2}}\right)^{\frac{\lambda a^{1/2}}{\pi}} \left(\frac{a-w}{w}\right)^{\frac{\lambda w^{1/2}}{\pi}} e^{i\lambda(w^{1/2}-a^{1/2})},$$

and its vector angle ranges from zero for $w = a$ to $-\lambda a^{1/2}$ for $w = 0$.

For w real and less than zero it takes the form

$$\left\{\frac{a^{1/2}+i(-w)^{1/2}}{a^{1/2}-i(-w)^{1/2}}\right\}^{\frac{\lambda a^{1/2}}{\pi}} \left(\frac{a-w}{-w}\right)^{\frac{\lambda i(-w)^{1/2}}{\pi}} e^{-i\lambda a^{1/2}},$$

which has modulus unity, and vector angle

$$\lambda\pi^{-1}\left\{2a^{1/2}\tan^{-1}\left(\frac{-w}{a}\right)^{1/2} + (-w)^{1/2}\log\left(\frac{a-w}{-w}\right) - \pi a^{1/2}\right\},$$

which ranges from $-\lambda a^{1/2}$ for $w = 0$ to zero for $w = -\infty$.

The assigning of different values to the parameter λ is equivalent to taking different powers of the factor corresponding to $\lambda = \pi$.

If $\mathscr{C}_{31}$ be employed in the transformation (95) the form of the curved part of the fixed boundary depends on both the parameters λ and p.

40. There is another method, slightly different from that just discussed, of building up a product whose limit is a curve-factor of a type suitable for dealing with problems of the class represented by fig. 14. In § 38 important factors of the product were curve-factors having all the same semi-infinite linear range of curvilinearity but different moduli; in the present instance curve-factors having different semi-infinite linear ranges, but the same modulus, are multiplied together.

A type of $\mathscr{C}_{13}$ is

$$\mathscr{C}_\kappa = \kappa^{1/2} + (w-a+\kappa)^{1/2},$$

which has the linear range $-\infty < w < a-\kappa$, angular range $\frac{1}{2}\pi$, and modulus, in the range, $(a-w)^{1/2}$.

Thus $\mathscr{C}_\kappa^{\,f(\kappa)\,d\kappa}$ has angular range $\frac{1}{2}\pi f(\kappa)\,d\kappa$ and modulus $(a-w)^{1/2 f(\kappa)\,d\kappa}$.

It follows that

$$\operatorname{Lim} \prod_{\kappa=\epsilon}^{\kappa=a} \{\kappa^{1/2} + (w-a+\kappa)^{1/2}\}^{f(\kappa)\,d\kappa}$$

has a double range of curvilinearity consisting of (i) $-\infty < w < 0$, on which the modulus is $\operatorname{Lim} \Pi\,(a-w)^{1/2 f(\kappa)\,d\kappa}$, and (ii) $0 < w < a-\epsilon$, on which the modulus has a less simple form. The angular range is $\operatorname{Lim} \frac{1}{2}\pi \Sigma f(\kappa)\,d\kappa$.

In general the limit form is

$$\mathscr{C}_{32} = \exp \int_\epsilon^a \log \{\kappa^{1/2} + (w-a+\kappa)^{1/2}\}\, f(\kappa)\,d\kappa, \quad . \quad . \quad . \quad . \quad . \quad (101)$$

with angular range $\frac{1}{2}\pi \int_\epsilon^a f(\kappa)\,d\kappa$ and modulus $(a-w)^{1/2 \int f(\kappa)\,d\kappa}$.

If an angular range $p\pi$ be desired f must be chosen so that $\frac{1}{2}\int_\epsilon^a f(\kappa)\,d\kappa = p$.

The transformation

$$dz = (w-a)^{-p}\,\mathscr{C}_{32}\,dw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

would give a configuration different from fig. 14 in this respect that the fixed boundary would be straight from $w = a$ to $w = a-\epsilon$, and would then round smoothly into a curve from $w = a-\epsilon$ to $w = 0$.

It is to be remarked that $\mathscr{C}_\kappa$ is not a curve-factor for $\kappa = 0$, though it is so for any positive value of κ however small. This might seem to preclude the putting of $\epsilon = 0$ in formula (101). However there is no real difficulty, for the subject of integration has no discontinuity at $\kappa = 0$, provided due precautions have been taken in the choice of f; and in any case the integral with lower limit zero is concerned with (amongst others) vanishingly small values of κ, but not with the actual value $\kappa = 0$. If the form of $\mathscr{C}_{32}$ for $\epsilon = 0$ be denoted by $\mathscr{C}_{33}$, the transformation

$$dz = (w-a)^{-p}\,\mathscr{C}_{33}\,dw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

gives a configuration like that of fig. 14.

A simple example is afforded by giving to $f(\kappa)$ the value $2pa^{-1}$. It is readily verified that

$$\int_\epsilon^a \log \{\kappa^{1/2} + (w-a+\kappa)^{1/2}\}\,d\kappa$$

$$= \Big[\{\kappa + \tfrac{1}{2}(w-a)\} \log \{\kappa^{1/2} + (w-a+\kappa)^{1/2}\} - \tfrac{1}{4}(w-a)\log(w-a) - \tfrac{1}{2}\kappa^{1/2}(w-a+\kappa)^{1/2}\Big]_\epsilon^a,$$

so that, denoting this particular case of $\mathscr{C}_{33}$ by $\mathscr{C}_{34}{}^{2p}$,

$$\mathscr{C}_{34} = (a^{1/2}+w^{1/2})^{1/2\left(\frac{a+w}{a}\right)}(w-a)^{1/4\left(\frac{w-a}{a}\right)}e^{-1/2a^{-1/2}w^{1/2}}. \quad . \quad . \quad . \quad . \quad . \quad (104)$$

By way of checking this result it is to be noticed that, for w real and $0 < w < a$, $\mathscr{C}_{34}$ takes the form

$$(a^{1/2}+w^{1/2})^{\frac{a+w}{2a}}(a-w)^{\frac{a-w}{4a}}e^{-1/2w^{1/2}a^{-1/2}}e^{i\pi\left(\frac{a-w}{4a}\right)},$$

in which the vector angle is $\frac{1}{4}\left(1-\dfrac{w}{a}\right)\pi$ and increases from zero to $\frac{1}{4}\pi$ as w diminishes from a to zero.

For w real and negative the form of $\mathscr{C}_{34}$ is

$$\{a^{1/2}+i(-w)^{1/2}\}^{\frac{a+w}{2a}}(a-w)^{\frac{a-w}{4a}}\exp i\left\{\pi\left(\frac{a-w}{4a}\right)-\frac{1}{2}\left(\frac{-w}{a}\right)^{1/2}\right\},$$

whose modulus is $(a-w)^{1/2}$, and vector angle χ, where

$$\chi = \frac{a+w}{2a}\tan^{-1}\left(\frac{-w}{a}\right)^{1/2}-\frac{1}{2}\left(\frac{-w}{a}\right)^{1/2}+\frac{1}{4}\pi\left(1-\frac{w}{a}\right);$$

it is readily seen that, as w decreases from zero to $-\infty$, χ passes from $\frac{1}{4}\pi$ to the limit value $\frac{1}{2}\pi$.

Thus $\mathscr{C}_{34}$ is a double curve-factor of angular range $\frac{1}{2}\pi$ equally divided between the two parts of its linear range $-\infty$ to 0, and 0 to a.

On the former part of the linear range the modulus is $(a-w)^{1/2}$, so that the transformation

$$dz = \mathrm{A}\,(w-a)^{-p}\,\mathscr{C}_{34}{}^{2p}\,dw \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

gives a configuration as in fig. 14, with a free stream-line tending to parallelism with the undisturbed stream.

In general

$$\int_{\epsilon}^{a}\log\{\kappa^{1/2}+(w-a+\kappa)^{1/2}\}\,\mathrm{F}'(\kappa)\,d\kappa = \Big[\mathrm{F}(\kappa)\log\{\kappa^{1/2}+(w-a+\kappa)^{1/2}\}\Big]_{\epsilon}^{a}-\frac{1}{2}\int_{\epsilon}^{a}\frac{\mathrm{F}(\kappa)\,d\kappa}{\kappa^{1/2}(w-a+\kappa)^{1/2}}.$$

An example, simpler than the former, is got by putting $\mathrm{F}(\kappa) = (\kappa/a)^{1/2}$, so that $f(\kappa) = \frac{1}{2}(\kappa a)^{-1/2}$.

This makes the integral with lower limit zero equal to

$$\log(a^{1/2}+w^{1/2})-a^{-1/2}\{w^{1/2}-(w-a)^{1/2}\},$$

and gives the curve-factor

$$\mathscr{C}_{35} = (a^{1/2}+w^{1/2})\exp\left[-a^{-1/2}\{w^{1/2}-(w-a)^{1/2}\}\right], \quad . \quad . \quad . \quad . \quad . \quad (106)$$

with the same sort of properties as $\mathscr{C}_{34}$.

$\mathscr{C}_{35}$ is really the product of two simpler curve-factors, one of the now familiar type of $\mathscr{C}_{13}$, the other

$$\mathscr{C}_{36} = \exp\left\{-\left(\frac{w}{a}\right)^{1/2}+\left(\frac{w}{a}-1\right)^{1/2}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

a double curve-factor whose linear ranges are $a > w > 0$ and $0 > w > -\infty$, having total angular range zero, and modulus unity for $w < 0$.

The formula of integration by parts suggests that other forms of $\mathscr{C}_{30}$ could be factorised in a corresponding manner.

41. A configuration of the type of fig. 15 is given by the transformation

$$dz = \frac{\text{A}\,(a^{1/2}+w^{1/2})^{2p-n}\,(b^{1/2}+w^{1/2})^{2q}\,\{(a-b)^{1/2}+(w-b)^{1/2}\}^{n}}{(w-a)^{p}\,(w-b)^{q}}\,dw, \quad . \quad . \quad . \quad . \quad (108)$$

where $a > b > 0$, n is arbitrary, and $p\pi$, $q\pi$ are the angles indicated in the figure. The boundary corresponding to w negative is a free stream-line. Greater generality

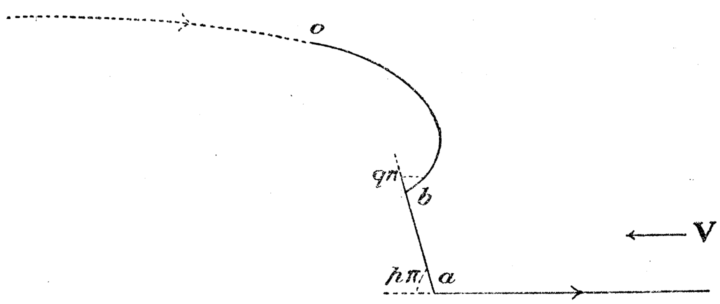

Fig. 15.

can be obtained by introducing factors of the type $\{(a-c)^{1/2}+(w-c)^{1/2}\}$, where $b > c > 0$.

General Remarks on Curve-factors.

42. *Note on the Relation between Angular Range and Order at Infinity.*—It has been already shown that the angular range of a curve-factor is π times its order at infinity; the same is obviously true for a Schwarzian factor. Hence all the transformations which have been considered are characterised by the property that the total angular range is π times the order at infinity. Thus if the angular range be $\beta-\pi$, where $2\pi \geqq \beta > 0$, the first approximation to the form of the transformation for values of w having very great modulus is

$$dz = \text{A}w^{\beta/\pi-1}\,dw,$$

where A is a real constant, and the limit form of the boundary consists of lines constituting the arms of an angle β.

Integration leads to the approximate relations

$$w = \mathrm{B}z^{\pi/\beta}+\mathrm{C}, \qquad dw/dz = \mathrm{B}\pi\beta^{-1}z^{\pi/\beta-1},$$

where B and C are real constants.

Now if w be expressible, for great values of z, as a series of terms of the type cz^{λ}, since w has to be real for z real, c must be real; and since w has to be real for z of the form $r\exp(i\beta)$, it is necessary that $\lambda\beta = n\pi$, where n is a positive or negative integer. Thus the admissible terms in the expansion are of the type $c_n z^{n\pi/\beta}$. When $\beta = \pi$ the expansion may also include a term in $\log z$, but it must be remembered that z, when great, is of a higher order of greatness than $\log z$.

The real part of w being represented by ϕ, it is known that

$$\iint \{(\partial\phi/\partial x)^2+(\partial\phi/\partial y)^2\}\, dx\, dy$$

taken over any area in the z plane is equal to $\int \phi\,(\partial\phi/\partial\nu)\, ds$ taken round the boundary, $\partial\nu$ representing an element of outward normal. If the boundary be made up of the locus of w real and an arc of a circle with centre at the origin and radius r, the subject of the line-integration is zero on the former part of the boundary. On the latter part $\partial\nu = \partial r$ and $ds = r\, d\theta$; thus

$$\iint \left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\} dx\, dy = \int_0^{\alpha} \phi\, \frac{\partial\phi}{\partial r}\, r\, d\theta,$$

where $\alpha \rightarrow \beta$ as r is increased indefinitely. If the area-integral is to cover the whole of the relevant region the limit of the right-hand side must be taken for $r \rightarrow \infty$.

A term $c_n z^{n\pi/\beta}$ in w would give in the limit of the line-integral a term $\frac{1}{2}n\pi c_n{}^2 r^{2n\pi/\beta}$, and this tends to zero if n is negative and to infinity if n is positive. If the area-integral vanishes w must be constant throughout the region. Hence, for an assigned value of β, there can be no w of any significance unless the most important term in w for z great has a positive index (or, as a possible alternative in the case of $\beta = \pi$, is a multiple of $\log z$). For this condition the least admissible value of n is unity.

Thus generally the transformations made up of curve-factors and Schwarzian factors are such that the most important term in w, for z great, is of the least possible order of magnitude that is consistent with w being other than a mere constant. It is true that, in the case of $\beta < \pi$, dw/dz is infinite for z infinite, but the conditions of the problem do not then admit of any w free from this objection.

In the hydrodynamical applications it may be said that the transformation gives, for any specified region in the z plane, an irrotational continuous motion which is as free from singularity at infinity as the nature of the geometrical configuration allows.

Adverting for example to § 14, it is seen now that the provisional exclusion of a negative value of $n\gamma$ was an unnecessary piece of caution. For $n\gamma$ negative the flow would take place in a region like that on the left of the arrows in fig. 16. It is true that in this flow the velocity at infinity would be infinite, but then there is no irrotational continuous flow possible in the region which does not have infinite velocity at infinity. The flow is, in fact, no more impossible than is the region in which it is supposed to take place.

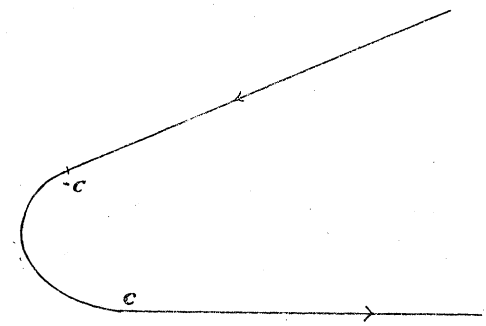

Fig. 16.

43. *Curve-factors not having a definite order at infinity.*—An example of a curve-factor of a kind not likely to be directly useful in physical applications is

$$\mathscr{C}_{37} = \exp\{f(w)+g(w)(w^2-c^2)^{1/2}\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

where f and g are rational algebraical functions, or any functions which have no infinities other than for w infinite. The vector angle, on the linear range $c > w > -c$, is $\chi = g(w)(c^2-w^2)^{1/2}$, and the angular range is zero.

$\mathscr{C}_{37}$ has usually no definite order at infinity, and so the proposition of §11 does not apply to it.

The possibility, however, of $\mathscr{C}_{37}$ having a definite order at infinity may be illustrated by an example which is in one respect a little more general than $\mathscr{C}_{37}$, namely,

$$\mathscr{C}_{38} = \exp\{w-c-(w-a)^{\alpha}(w-b)^{1-\alpha}\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

For w great this tends to the limit $\exp\{\alpha a+(1-\alpha)b-c\}$, so that the order at infinity is zero.

Of somewhat similar character are the curve-factors

$$\mathscr{C}_{39} = \exp\{(w-a)^{\alpha}(w-b)^{1-\alpha}-(w-a)^{\beta}(w-b)^{1-\beta}\}, \quad . \quad . \quad . \quad . \quad . \quad (111)$$

$$\mathscr{C}_{40} = \exp[w^2-\{\alpha a+(1-\alpha)b+c\}w-(w-c)(w-a)^{\alpha}(w-b)^{1-\alpha}]. \quad . \quad (112)$$

Curve-factors Regarded as the Limits of Products of Schwarzian Factors.

44. The Schwarz-Christoffel transformation being so widely known and of such proved utility, the most natural way of trying to obtain a transformation for the conformal representation of a region whose boundary is partly curvilinear would seem to be to treat the curve as the limit of a rectilineal polygon and to seek the corresponding limit of the product of the Schwarzian factors appropriate to the corners that have to be smoothed out into a continuous curve.

If χ be, as usual, the angle between the axis of z real and the tangent at any point of the curve corresponding to w real, the sense of the tangent being that of w increasing, the external angle of the polygon whose limit is to be the curve may be taken to be $d\chi$, so that, if θ be the value of w at the corner, the Schwarzian factor is $(w-\theta)^{-d\chi/\pi}$. The parameter θ varies continuously along the curve, and the curve-factor

$$\lim \Pi \, (w-\theta)^{-d\chi/\pi}$$

can generally be replaced by the expression

$$\mathscr{C}_{41} = \exp \int \log (w-\theta) \, (-d\chi/\pi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

the integral extending over the whole curve.

It is to be noticed that in this expression there may be an infinity in the subject of integration at a point corresponding to a real value of w. This infinity, being less powerful than an infinity of the type $(w-\theta)^{-1}$, does not interfere with absolute convergence of the integral. Formal proof of this statement is considered unnecessary here, being of a character readily suggested by familiar treatments of convergence tests. It may be illustrated, in the hydro-dynamical application, by the fact that, while a convex angle in the boundary gives rise to an infinite velocity, a convex smooth curve does not.

But, while there is no divergence of the integral, the formula (113) is nevertheless indefinite in the absence of any specified functional relation between the variables θ and χ. The Schwarz transformation takes account only of the angles of the configuration to which it is applied, and leaves the adjustments of all lengths to be dealt with after transformation by the assigning of suitable values to the various parameters associated with w in the factors. It is therefore not surprising that a transformation containing only Schwarzian factors and limits of products of such factors should ensure only that the directional and angular aspects of the configuration are properly dealt with, leaving all settlement of correct linear dimensions to be adjusted subsequently by the assigning of suitable values to isolated parameters and suitable functional relations to parameters which vary continuously.

It is therefore proper to assume a functional relation between χ and θ, say $\chi = \pi f(\theta)$, with the reservation that the nature of the function f is determined by the configuration which is being dealt with, and depends not merely on the curve to which $\mathscr{C}_{41}$ corresponds but on the whole angular and linear configuration of the prescribed boundary.

45. Before attempting to formulate the condition which f must satisfy in order that $\mathscr{C}_{41}$ in a transformation of suitable type may represent a curve of assigned form, it is convenient to note the usefulness of an alternative method of applying formula (113) which consists in assigning such arbitrary forms to f as lead to simple

integrations, and so generating curve-factors whose geometrical significance in typical transformations can be examined. As the geometrical interpretation has to do with real values of w, it is to be remarked that for w real and $b > w > a$

$$-\int_a^b \log(w-\theta) f'(\theta)\, d\theta = -\int_a^b \log|w-\theta| f'(\theta)\, d\theta + i\pi\{f(w)-f(b)\}, \quad . \quad (114)$$

so that the vector angle of $\mathscr{C}_{41}$ is $\pi\{f(w)-f(b)\}$ and its modulus

$$\exp\left\{-\int_a^b \log|w-\theta| f'(\theta)\, d\theta\right\}.$$

A simple example is got by taking $f(\theta) = -p\theta+q$, where p and q are constants, so that $d\chi = -\pi p d\theta$. The argument of the exponential in the curve-factor is

$$p\int_a^b \log(w-\theta)\, d\theta = p\Big[(w-\theta)\{1-\log(w-\theta)\}\Big]_a^b$$

$$= p\{a-b+(w-a)\log(w-a)-(w-b)\log(w-b)\}.$$

After omission of a constant factor this yields the curve-factor

$$\mathscr{C}_{42} = (w-a)^{p(w-a)}(w-b)^{-p(w-b)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

It is readily seen that the angular range of $\mathscr{C}_{42}$ is $\pi p(b-a)$, and that for w great it approximates to the form $e^{(b-a)} w^{p(b-a)}$, so that there is a definite order at infinity, namely $p(b-a)$. The curve-factor does not tend to zero as w approaches a or b.

By putting $f(\theta) = \theta^2$ one gets the curve-factor

$$\mathscr{C}_{43} = (w-b)^{(b^2-w^2)}(w-a)^{(w^2-a^2)} e^{1/2(a-b)(a+b+2w)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

with angular range $\pi(b^2-a^2)$ and order at infinity (b^2-a^2).

There would probably be little difficulty in finding a considerable number of forms of $f(\theta)$, which would permit of evaluation of the integral in formula (113) and so yield types of curve-factor.

46. The problem of formulating the condition which $f(\theta)$ must satisfy, in order that the resulting curve-factor (in combination with suitable Schwarzian factors) may be applicable to a prescribed curve, may be exemplified by taking the case of the doubly-pointed ship. Suppose a configuration like fig. 3 has to be dealt with, the form of the curve in the linear range $-c < w < c$ being prescribed. The transformation will be of the form

$$dz = (w-c)^{-p}(w+c)^{-q} \exp\left\{-\int_{-c}^{c} f'(\theta)\log(w-\theta)\, d\theta\right\} dw, \quad . \quad . \quad . \quad (117)$$

and the question is what is required of $f(\theta)$ in order that the curve corresponding to $-c < w < c$ may be as prescribed.

It seems simplest to deal with the curvature of the curve. When dz is put equal to $ds \exp(i\chi)$, it appears that, for w real and between $-c$ and c,

$$\begin{aligned}\chi &= -p\pi + \pi\{f(w) - f(c)\} \\ &= \pi f(w), \quad \ldots \ldots \ldots \quad (118)\end{aligned}$$

the assumption $-p\pi = \pi f(c)$ being equivalent to the previously implied assumption that the curve-factor of the type $\mathscr{C}_{41}$ has no latent Schwarzian singularity at either end of the range, so that the transformation $dz = \mathscr{C}_{41}\, dw$ would give a boundary without corners. It appears also that

$$ds = (c-w)^{-p}(w+c)^{-q} \exp\left\{-\int_{-c}^{c} f'(\theta) \log|w-\theta|\, d\theta\right\} dw, \quad \ldots \quad (119)$$

so that

$$\frac{ds}{d\chi} = \frac{\exp\left\{-\int_{-c}^{c} f'(\theta) \log|w-\theta|\, d\theta\right\}}{\pi f'(w)\,(c-w)^{p}\,(w+c)^{q}} \quad \ldots \ldots \ldots \quad (120)$$

Now on the prescribed curve $ds/d\chi$ is a known function of χ, say $\mathrm{R}(\chi/\pi)$; hence f must satisfy

$$\mathrm{R}\{f(w)\} = \frac{\exp\left\{-\int_{-c}^{c} f'(\theta) \log|w-\theta|\, d\theta\right\}}{\pi f'(w)\,(c-w)^{p}\,(w+c)^{q}}, \quad \ldots \ldots \quad (121)$$

which can be regarded as an integral equation in f'.

A less complicated-looking form of the condition can be got by taking logarithms of both sides of (121) and differentiating with respect to w. In carrying out this operation it is useful to note that the differentiation of the definite integral is effected by differentiation under the sign of integration and taking the Cauchy principal value (indicated by P.V.) of the resulting integral; proof of this is omitted as the result is probably well known. The resulting expression of the condition which f must satisfy is

$$f'(w)\frac{\mathrm{R}'\{f(w)\}}{\mathrm{R}\{f(w)\}} + \frac{f''(w)}{f'(w)} - \frac{p}{c-w} + \frac{q}{w+c} = -\text{P.V.}\int_{-c}^{c} \frac{f'(\theta)}{w-\theta}\, d\theta. \quad \ldots \quad (122)$$

47. *Transformation for any Prescribed Boundary.*—Instead of confining the application of the method of § 44 to a single curved portion of the boundary of the region which is to be represented conformally on the half-plane of w, it is legitimate to deal with the whole boundary in a single formula, namely

$$dz = \exp\left\{-\frac{1}{\pi}\int \log(w-\theta)\, d\chi\right\} dw, \quad \ldots \ldots \ldots \quad (123)$$

where θ ranges from $+\infty$ to $-\infty$, and there is an unknown and possibly discontinuous, but definite, functional relation between θ and χ. A straight part of the boundary, along which $d\chi$ is zero, makes no contribution to the formula. A corner, where χ undergoes finite change, say $p\pi$, while θ is at a standstill say at α, contributes to the integral an amount $-p\pi \log(w-\alpha)$, and so leads to the ordinary Schwarzian or corner factor. The curved portions of the boundary contribute distinct curve-factors of the type $\mathscr{C}_{41}$.

For any one continuously curved stretch of the boundary, say $a < w < b$, let $f_1(\theta)$ be the suitable expression for χ/π, and let $F_1(w)$ be the modulus of all that part of the right-hand side of formula (123) which corresponds to values of θ outside the range $a < \theta < b$. Then the condition that f_1 has to satisfy is of the type

$$R_1\{f_1(w)\} = \frac{F_1(w)}{\pi f_1'(w)} \exp\left\{-\int_a^b f_1(\theta) \log|w-\theta|\, d\theta\right\}. \quad . \quad . \quad . \quad (124)$$

To other curved parts of the boundary correspond other functions, $f_2, f_3, \ldots$, &c., which satisfy similar conditions involving $F_2, F_3, \ldots$, &c., but F_1 depends on $f_2, f_3, \ldots$, and F_2 depends on $f_1, f_3, \ldots$, so that the complete expression of the conditions which the f's must satisfy is a complicated set of simultaneous integral equations with the function $\log|w-\theta|$ as kernel.

48. *Free Stream-lines.*—As an alternative to the formulation of the previous article it may be assumed that along the curve $\chi = \pi\psi(s)$, and $s = g(w)$, where g is the unknown function. Then

$$g'(w)\, dw = ds = F(w) \exp\left\{-\int_a^b \psi'\{g(\theta)\} g'(\theta) \log|w-\theta|\, d\theta\right\} dw,$$

so that

$$\log\left\{\frac{g'(w)}{F(w)}\right\} + \int_a^b \psi'\{g(\theta)\} g'(\theta) \log|w-\theta|\, d\theta = 0, \quad . \quad . \quad . \quad . \quad (125)$$

an integral equation (or equations) in g'.

The characteristic property of a free stream-line is that along it the velocity is constant, say unity, so that $w = s$ and $g' = 1$. Thus if one portion of the boundary, instead of being of prescribed shape, is to be a free stream-line, the corresponding function ψ must satisfy the condition

$$-\log F(w) + \int_a^b \psi'(\theta) \log|w-\theta|\, d\theta = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

an integral equation in ψ' with kernel $\log|w-\theta|$.

The solution of this equation for the case in which the fixed boundary is a rectilineal polygon is derivable from the results of §37.

Transformations Involving Both Variables Explicitly.

49. The study of conformal representation by means of transformations of the type

$$f(z)\, dz = F(w)\, dw$$

would obviously present even greater difficulty than the kind of transformation hitherto considered; but in connexion with the problem typified in fig. 3, the problem namely of obtaining shapes of doubly-pointed ships whose hydrodynamical effect in longitudinal motion can be exactly specified, something can be done with the more general transformation.

It is easy to verify that the transformation for a semi-circle and the productions of a diameter, namely

$$dz = \mathrm{C}\,(w^2-c^2)^{-1/2}\,\{w+(w^2-c^2)^{1/2}\}\,dw,$$

is equivalent to

$$\frac{dz}{z} = d\zeta = \frac{dw}{(w^2-c^2)^{1/2}}; \quad \ldots \ldots \ldots \quad (127)$$

and that the transformation for a semi-ellipse, $(\mathrm{C}c\cosh\alpha,\ \mathrm{C}c\sinh\alpha)$, and the productions of its principal diameter, namely

$$dz = \mathrm{C}\,(w^2-c^2)^{-1/2}\,\{w\sinh\alpha+(w^2-c^2)^{1/2}\cosh\alpha\}\,dw,$$

is equivalent to

$$\frac{dz}{(z^2-\mathrm{C}^2c^2)^{1/2}} = d\zeta = \frac{dw}{(w^2-c^2)^{1/2}}, \quad \ldots \ldots \ldots \quad (128)$$

an intermediate variable ζ being for convenience introduced in each case.

In these forms it is to be noticed: (i) that the factors in the denominators on the left-hand side do not lead to zeros of dz/dw or to corners because the points in the z plane where they have zeros are not in the relevant region, and (ii) that the only factors left on the right-hand side are the Schwarzian or corner factors. Thus the effect of the (z, ζ) transformation is to straighten out the curved part of the boundary without alteration of the corner-angles, and the effect of the (ζ, w) transformation is to smooth out the corners.

In both cases the (z, ζ) transformation is of the Schwarzian type, making the axis of z real correspond to a rectilineal open polygon in the ζ plane, but these polygons do not constitute the boundaries of the regions which are relevant to the present problem. A straight line joining points in the first and last arms of the open polygon in the ζ plane screens off all the original corners from the relevant region, as it were short-circuiting a part of the boundary including all the corners; and the corresponding line in the z plane is a curved line which screens off from the relevant region all those points on the real axis which corresponded to the corners of the Schwarzian transformation.

This aspect of the (z, ζ) transformation suggests generalisation by the introduction of any number of corners on the broken line in the ζ plane which corresponds to the part of the axis of z real which is screened off from the relevant region.

The desired configuration in the z plane being as shown in fig. 17, the values $p\pi$, $q\pi$ of the marked angles being prescribed, and the values c, $-c$ being assigned to z at the ends of the curve, any number of real parameters a_1, a_2, a_3, ... may be taken, subject only to the condition $c > a_1 > a_2 > a_3 > \ldots > -c$, and another set of parameters κ_1, κ_2, κ_3, ... associated with them, subject to the condition

$$\kappa_1 + \kappa_2 + \kappa_3 + \ldots = p + q. \quad \ldots \ldots \quad (129)$$

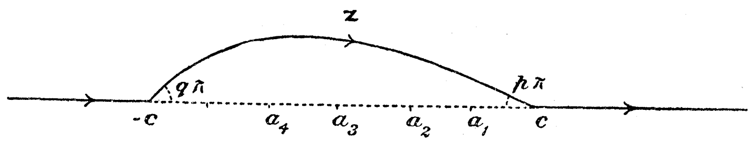

Fig. 17.

Then the transformation

$$\frac{dz}{(z-a_1)^{\kappa_1}(z-a_2)^{\kappa_2}(z-a_3)^{\kappa_3}\ldots} = d\zeta \quad \ldots \ldots \quad (130)$$

makes the axis of z real correspond to an open polygon in the ζ plane such as that in fig. 18. The external angles $\kappa_1\pi$, $\kappa_2\pi$, ..., so that the angle between the directions of the arrows on the first and last sides is $(\kappa_1 + \kappa_2 + \ldots)\pi$. The straight line in the ζ plane joining the points $\pm c$ must correspond to a convex curve in the z plane.

The representation of the ζ configuration upon the half plane of w is of the Schwarzian type, taking account of the corners in the boundary of the relevant

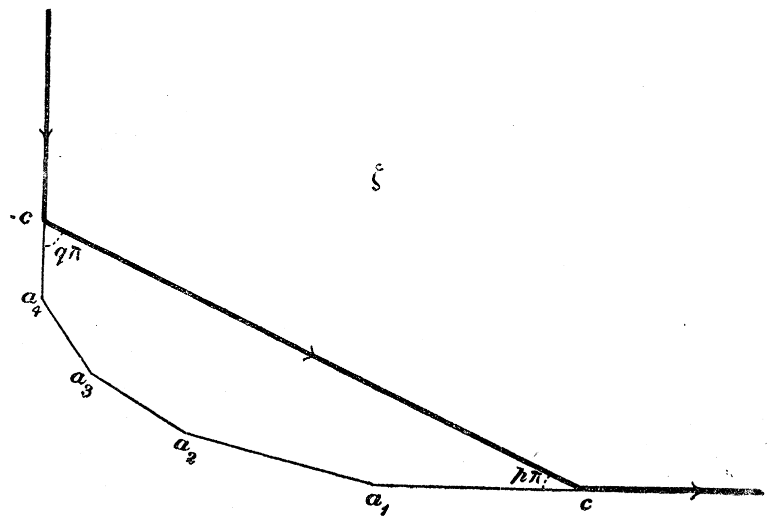

Fig. 18.

region. Thus if α, β be the values of w corresponding to the corners, the full (z, w) transformation is

$$\frac{dz}{(z-a_1)^{\kappa_1}(z-a_2)^{\kappa_2}(z-a_3)^{\kappa_3}\ldots} = \frac{dw}{(w-\alpha)^p(w-\beta)^q}, \quad \ldots \ldots \quad (131)$$

subject to the condition (129).

Still further generalisation can be secured by making some of the screened-off sides of the polygon in the ζ plane curved. This can be done by introducing curve-factors (expressed now in terms of z) into the left side of the transformation. For example if one factor $\mathscr{C}$ of linear range a_r to a_{r+1} and angular range $\gamma\pi$ were introduced, the transformation would be of the type

$$\mathscr{C}\Pi (z-a)^{-\kappa}\, dz = (w-\alpha)^{-p}(w-\beta)^{-q}\, dw, \quad \ldots \ldots \quad (132)$$

with the condition

$$\kappa_1+\kappa_2+\ldots-\gamma = p+q. \quad \ldots \ldots \quad (133)$$

A practical inconvenience of the present method of constructing transformations applicable to the ship problem lies in the fact that, while the condition (129) or alternatively (133) secures that the angles $p\pi$ and $q\pi$ shall have any prescribed sum, it does not suffice to secure prescribed separate values for these angles. When this is desired there is a further condition to be satisfied in the form of a relation between the parameters $-c$, a_1, a_2, ..., c. What is wanted is that the vector angle of the complex $\zeta(c)-\zeta(-c)$ shall be $-p\pi$, and this is equivalent to

$$\text{Vector angle of} \int_{-c}^{c} f(z)\, dz = -p\pi.$$

Generally the fulfilment of this condition cannot be arranged for without previous evaluation of the indefinite integral of $f(z)$. Thus the problem of integration, which must inevitably be faced in any case in the detailed interpretation of a transformation, presents itself here at an earlier stage.

50. The method of the previous article may be further modified by associating with dz such Schwarzian factors, powers of $z\pm c$, as shall remove the corners at $\pm c$ and make the first and last portions of the boundary productions of the straight line that corresponds to the curve in the z plane. In this case there are no Schwarzian factors to be associated with dw, so ζ and w become identical save for a constant

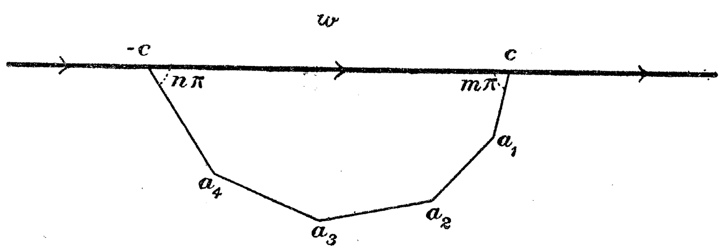

Fig. 19.

multiplier, and the configuration in the w plane is as shown in fig. 19, the marked angles being called $m\pi$ and $n\pi$. The transformation is of the form

$$\frac{(z-c)^m (z+c)^n\, dz}{(z-a_1)^{\kappa_1}(z-a_2)^{\kappa_2}\ldots} = \mathrm{C}\, dw.$$

In comparing the z and w configurations as represented by figs. 17 and 19, it is to be noted that corresponding angles at c, though not equal, are proportional. Hence

$m\pi : \pi = p\pi : (1-p)\pi$, so that $m = p/(1-p)$; similarly $n = q/(1-q)$. Thus the transformation is

$$\frac{(z-c)^{\frac{p}{1-p}}(z+c)^{\frac{q}{1-q}}\,dz}{(z-a_1)^{\kappa_1}(z-a_2)^{\kappa_2}(z-a_3)^{\kappa_3}\ldots} = \mathrm{C}\,dw, \qquad (134)$$

with the conditions

$$c > a_1 > a_2 > a_3 \ldots > -c, \qquad (135)$$

and

$$\kappa_1+\kappa_2+\kappa_3+\ldots = \frac{p}{1-p}+\frac{q}{1-q}, \qquad (136)$$

the latter being the condition that the transformation have zero angular range. In the application to ships there must also be the condition for vanishing of the source term in the expansion of w for z great.

Clearly it is possible, with obvious precautions, to introduce on the left-hand side of (134) curve-factors in z having a linear range inside the range $-c$ to c, due account being taken of their angular range in (136).

A particular example (as regards the w configuration rather a limit case) of the transformation (134) is got by putting $p = \frac{1}{2}$, $q = \frac{1}{2}$, $\kappa_1 = 1$, $\kappa_2 = 1$, $a_1 = a$, $a_2 = -a$. This gives

$$\mathrm{C}\,dw = \frac{z^2-c^2}{z^2-a^2}\,dz, \qquad (137)$$

wherein $c > a$. The curves in the z plane corresponding to this transformation are RANKINE'S "oval neoïds."*

51. The transformation for the semi-ellipse ($c \cosh \alpha$, $c \sinh \alpha$) and the productions of its minor axis, written in the form

$$\frac{(z^2-c^2\sinh^2\alpha)\,dz}{(z^2+c^2)^{1/2}\{z\cosh\alpha+(z^2+c^2)^{1/2}\sinh\alpha\}} = dw, \qquad (138)$$

exemplifies the possible presence in the denominator on the left-hand side of (i) a function having imaginary zeros which are not in the relevant region, (ii) a function entirely free from zeros, and possessed of branch points outside the relevant region. Doubtless it would be possible to effect some generalisation of a formula including these features.

SUPPLEMENTARY NOTE ON CURVE-FACTORS.

(*Added* 15*th June*, 1915.)

52. It should be noticed that the product of two curve-factors, of which the linear range of one is contained in the linear range of the other, is itself a curve-factor for the greater linear range. The resultant curve-factor is not, however, simple in the sense specified in § 4, for though a curve represented by the curve-factor in any

* RANKINE, "On Plane Water-Lines in Two Dimensions," 'Scientific Papers,' p. 495.

transformation might have no discontinuity of direction in the full linear range there would be discontinuities in the analytical form of the equation to the curve at the extremities of the inner range.

This illustrates the fact that a curve-factor, which, for its total linear range of curvilinearity, is not necessarily simple, may have branch-points not only at the extremities of the linear range but also at points within the range. If the general character of the types already chiefly considered is to be maintained in such a multiple curve-factor, the net effect of proceeding along the real axis of w with a suitable detour round each branch-point must be simply a change of sign when the whole range has been traversed. Thus the formula

$$\mathscr{C}_{44} = w-k+(w-a)^p(w-c_1)^{n_1}(w-c_2)^{n_2}\dots(w-b)^q, \quad . \quad . \quad . \quad . \quad (138\text{A})$$

where $a > c_1 > c_2 > \dots > c_r > k > c_{r+1} > \dots > b$, gives a curve-factor provided all the indices are positive, and

$$p+n_1+n_2+\dots+q = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

The function has no real zeros, and there can be no imaginary zeros in the relevant region unless the angular range is different from π. On writing down the expression for the tangent of the vector angle χ in any sub-range c_s to c_{s+1} it can be verified that, except in the case of $s = r$, infinities of $\tan\chi$, if they occur at all, must occur in pairs which are not separated by a zero. Thus all the sub-ranges have zero angular ranges, except one whose angular range is π, and so the total angular range is π.

The complete curve corresponding, in any transformation, to a factor $\mathscr{C}_{44}{}^m$ consists of two undulatory curves of zero angular range extending from a to c_r and from c_{r+1} to b, joined by a curve of angular range $m\pi$ corresponding to the interval c_r to c_{r+1}.

Of course no one of the c's may be equal to k, as if $w-k$ were a factor of $\mathscr{C}_{44}$ there would be a corner at $w = k$.

53. The multiple character of the curve-factor $\mathscr{C}_{44}$ arises from the discrete distribution along the linear range of branch-points of definite order. This feature can be eliminated by substituting a continuous distribution of branch-points, each factor having an infinitesimal index except the factors at the ends of the range. Thus to the factor $w-\theta$ is assigned the index $f(\theta)\,d\theta$, and the limit is taken for vanishing of $d\theta$ in each case. The form thus suggested is

$$\mathscr{C}_{45} = w-k+(w-a)^p(w-b)^q \exp\int_b^a f(\theta)\log(w-\theta)\,d\theta. \quad . \quad . \quad . \quad (140)$$

So long as $f(\theta)$ is free from infinities in the linear range, the second term of this expression has no zeros in that range, and it will be supposed that $f(\theta)$ is thus restricted. An infinity of $f(\theta)$ at $\theta = k$ might introduce a power of $w-k$ as a factor in $\mathscr{C}_{45}$ and so introduce a corner; infinities of $f(\theta)$ at other points of the range, if their effect were to introduce factors of the type $(w-c)^n$ into the second term, would leave $\mathscr{C}_{45}$ still a curve-factor but not a simple curve-factor.

The undulations which characterise $\mathscr{C}_{44}$ do not appear in curves represented by $\mathscr{C}_{45}$. The relation corresponding in this case to formula (139) is

$$p+q+\int_b^a f(\theta)\,d\theta = 1. \qquad (141)$$

An example of $\mathscr{C}_{45}$, got by putting $f(\theta)$ equal to a constant γ, is

$$\mathscr{C}_{46} = w-k+(w-a)^{p-\gamma(w-a)}(w-b)^{q+\gamma(w-b)}e^{-\gamma(a-b)}, \qquad (142)$$

subject to

$$p+q+\gamma(a-b) = 1. \qquad (143)$$

54. A generalisation of $\mathscr{C}_{44}$ is

$$\mathscr{C}_{47} = (w-k_1)^{m_1}(w-k_2)^{m_2}\ldots+(w-a)^p(w-c_1)^{n_1}(w-c_2)^{n_2}\ldots(w-b)^q, \qquad (144)$$

where the k's lie between a and b but do not coincide with any of the c's, the m's are all positive, and

$$m_1+m_2+\ldots = 1. \qquad (145)$$

It is readily verified that, for real values of w between a and b, the imaginary part of $\mathscr{C}_{47}$ does not vanish, while the real part vanishes an odd number of times. Hence the angular amplitude is π.

Treatment on the lines of article 53 leads to the form

$$\mathscr{C}_{48} = \exp\int_b^a \mathrm{F}(\theta)\log(w-\theta)\,d\theta+\exp\int_b^a f(\theta)\log(w-\theta)\,d\theta, \qquad (146)$$

where F and f are functions of θ free from infinities between a and b.

55. The forms obtained in the last three articles suggest the possibility that the addition of Schwarzian factors or the products of Schwarzian factors to one another, or of curve-factors to one another, or of members of the one class to members of the other class, may generally lead to results which are themselves curve-factors, provided all the terms so added have the same angular range. Probably further limitations would have to be introduced into the enunciation of such a theorem to make it valid, but it seems clear that the method of addition is a useful means of obtaining fresh forms of curve-factor.

The following examples suggest themselves:—

$$\mathscr{C}_{49} = (w-a)^m+(w-c_1)^m+(w-c_2)^m+\ldots+(w-b)^m, \qquad (147)$$

where $1 > m > 0$, and $a > c_1 > c_2 > \ldots > b$; and

$$\mathscr{C}_{50} = \int_b^a f(\theta)(w-\theta)^m\,d\theta, \qquad (148)$$

where $f(\vartheta)$ is positive for values of θ between a and b.